PART - A

Unit - III

DIFFERENTIAL CALCULUS - 3

3.1 Partial Differentiation

3.11 Introduction

So far we have discussed various concepts related to a function of one independent variable only. That is, discussion related to $y = f(x)$.

Practically we do come across with quantities whose value depend on more than one independent variable.

For example the area (A) of a rectangle depends on its length (l) and breadth (b). That is $A = f(l, b)$. Volume (V) of a parallelopiped depends on its length (l), breadth (b) and height (h). That is $V = f(l, b, h)$.

We are conversant with the differentiation of a function of one independent variable and this topic is again the discussion of known concepts of differentiation in a broader perspective.

Precisely, this topic deals with the differentiation of a function of many independent variables.

3.12 Partial derivatives

Let u be a function of two independent variables x and y.

ie., $u = f(x, y)$.

Recapitulating the definition of $\frac{dy}{dx}$ in the case of $y = f(x)$,

[*ordinary derivative to be specific*] the partial derivatives of u *w.r.t* x and also *w.r.t* y are defined in a similar fashion.

Here goes the definitions.

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x} \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \qquad \ldots (2)$$

["∂" *is the symbol of partial derivative usually read as "del" and it should not be written as "δ" (delta) an acquainted symbol*]

Thus (1) is *called the partial derivative of* u *w.r.t* x and (2) is called the partial derivative of u *w.r.t* y.

It should be observed that in the equation (1) x is varying whereas y is remaining constant. Also in the equation (2) y is varying whereas x is remaining constant.

The derivative of u *w.r.t* x treating y as a constant is called as the partial derivative of u *w.r.t* x and is denoted by $\frac{\partial u}{\partial x}$ or u_x

Similarly the derivative of u *w.r.t* y treating x as a constant is called the partial derivative of u *w.r.t* y and is denoted by $\frac{\partial u}{\partial y}$ or u_y

The product rule, quotient rule and the function of a function rule continues to hold good here also.

Observe the following comparisons.

Ordinary derivative	**Partial derivatives**
1. $y = 3x^2 + 6x + 7$ $\frac{dy}{dx} = 6x + 6 + 0 = 6x + 6$	1. $u = 3x^2 y + 6xy^2 + 7$ $\frac{\partial u}{\partial x} = (6x)\,y + 6 \cdot 1 \cdot y^2 + 0 = 6xy + 6y^2$ [y is treated as constant] $\frac{\partial u}{\partial y} = 3x^2 \cdot 1 + 6x \cdot 2y + 0 = 3x^2 + 12xy$ [x is treated as constant]
2. $y = e^{4x+3}$ $\frac{dy}{dx} = e^{4x+3} \cdot \frac{d}{dx}(4x+3)$ $= e^{4x+3} \cdot 4 = 4e^{4x+3}$	2. $u = e^{4x+3y}$ $\frac{\partial u}{\partial x} = e^{4x+3y} \cdot \frac{\partial}{\partial x}(4x+3y)$ $= e^{4x+3y} \cdot (4+0) = 4e^{4x+3y}$ $\frac{\partial u}{\partial y} = e^{4x+3y} \cdot \frac{\partial}{\partial y}(4x+3y)$ $= e^{4x+3y} \cdot (0+3) = 3e^{4x+3y}$
3. $y = \sin 5x$ $\frac{dy}{dx} = \cos 5x \cdot 5 = 5 \cos 5x$	3. $u = \sin(xy)$ $\frac{\partial u}{\partial x} = \cos(xy)\frac{\partial}{\partial x}(xy) = \cos(xy) \cdot y$ $\frac{\partial u}{\partial y} = \cos(xy)\frac{\partial}{\partial y}(xy) = \cos(xy) \cdot x$
4. $y = \tan^{-1}(2/x)$ $\frac{dy}{dx} = \frac{1}{1+(2/x)^2} \cdot \frac{d}{dx}\left(\frac{2}{x}\right)$ $= \frac{x^2}{x^2+4} \cdot -\frac{2}{x^2} = -\frac{2}{x^2+4}$	4. $u = \tan^{-1}(y/x)$, $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\ \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$ *ie.,* $= \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$ $\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\ \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$ $= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

Ordinary derivative	Partial derivatives
5. $\frac{d}{dx}[f(y)]=f'(y)\frac{dy}{dx}$	5. If r is a function of x and y, $\frac{\partial}{\partial x}[f(r)]=f'(r)\frac{\partial r}{\partial x}$, $\frac{\partial}{\partial y}[f(r)]=f'(r)\frac{\partial r}{\partial y}$

General principle of partial differentiation

Given a function of many independent variables, the derivative of this function with respect to a particular independent variable, keeping (*treating*) all other independent variables as constants is the general principle of partial differentiation.

Higher order partial derivatives

These are also analogous with the higher order ordinary derivatives.

Let us suppose that $u = f(x, y)$. The development of higher order partial derivatives is as exhibited below.

$$u = f(x, y)$$

First order partial derivatives

$$u_x = \frac{\partial u}{\partial x} \qquad\qquad u_y = \frac{\partial u}{\partial y}$$

Second order partial derivatives

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad\qquad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \quad \longleftarrow \text{Mixed partial derivatives} \longrightarrow \quad u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y}$$

⟶ and so-on ⟵

It is very important to note that;

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} \quad \text{or} \quad u_{yx} = u_{xy}$$

WORKED PROBLEMS

[illegible] partial derivatives

Given an explicit function of more than one independent variable we find the required partial derivatives just keeping in mind the *general principle of partial differentiation* stated earlier along with the well acquainted rules of differentiation. The following note on symmetric functions will be highly useful for certain problems.

Note : Symmetric function : *A function $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x)$ and a function $f(x, y, z)$ is said to be symmetric if $f(x, y, z) = f(y, z, x) = f(z, x, y)$. In general we can say that a function of several variables is symmetric if the function remains unchanged (invariant) when the variables are cyclically rotated. Observe the following examples.*

(i) $x + y,\ x^2 + y^2,\ \dfrac{x^2 + y^2}{x + y},\ x^2 + xy + y^2,\ \log\sqrt{x^2 + y^2}$ *etc.*

are symmetric functions of two variables as it can be easily seen that when x is replaced by y and y by x the functions remain the same.

(ii) $x^2 + y^2 + z^2,\ xy + yz + zx,\ x/y + y/z + z/x,\ \log(x + y + z),$

$x^3 + y^3 + z^3 - 3xyz$ *etc.*

are symmetric functions of three variables.

It is very important to note that, *if we have a symmetric function of three variables say $u = f(x, y, z)$ then by just computing u_x or u_{xx} or u_{xy} we can simply writedown easily the other partial derivatives (u_y, u_z) or (u_{yy}, u_{zz}) or (u_{yz}, u_{zx}) by simple guess work. There is no need to show the working of similar computation of partial derivatives*

We have $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

$\therefore \quad \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y$ ... (*y is treated as constant.*)

Differentiating this *w.r.t* x partially again,

$$\frac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y \quad \text{... (Again } y \text{ is treated as constant.)}$$

Next, $\dfrac{\partial u}{\partial y} = -6xy - e^x \sin y$ ... (*x is treated as constant.*)

Differentiating this *w.r.t* y partially again,

$$\frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \quad \text{... (Again } x \text{ is treated as constant.)}$$

Now $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y = 0$

This proves the desired result.

2. If $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

>> We have $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$

$\therefore \quad \frac{\partial u}{\partial x} = e^{-2\pi^2 t} (\pi \cos \pi x) \sin \pi y \quad \ldots$ *(t & y are treated as constants)*

Differentiating this *w.r.t* x partially again,

$$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t} (-\pi^2 \sin \pi x) \sin \pi y = -\pi^2 u$$

$\frac{\partial u}{\partial y} = e^{-2\pi^2 t} \sin \pi x (\pi \cos \pi y) \quad \ldots$ *(t & x are treated as constants)*

Differentiating this *w.r.t* y partially again,

$$\frac{\partial^2 u}{\partial y^2} = e^{-2\pi^2 t} \sin \pi x (-\pi^2 \sin \pi y) = -\pi^2 u$$

Thus $\quad \text{L.H.S} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi^2 u - \pi^2 u = -2\pi^2 u \quad \ldots(1)$

Also $\quad \frac{\partial u}{\partial t} = e^{-2\pi^2 t} (-2\pi^2) \sin \pi x \sin \pi y \quad \ldots$ *(x & y are treated as constants)*

Thus $\quad \text{R.H.S} = \frac{\partial u}{\partial t} = -2\pi^2 u \quad \ldots(2)$

$\therefore$ from (1) and (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

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3. *If* $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ *show that* $x u_x + y u_y = 1$

>> $\quad u = \log(x^2 + y^2) - \log(x + y)$

$\therefore \quad u_x = \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{x + y} \cdot 1$

and $\quad u_y = \frac{1}{x^2 + y^2} \cdot 2y - \frac{1}{x + y} \cdot 1$

Now, $x u_x + y u_y = \dfrac{2x^2}{x^2+y^2} - \dfrac{x}{x+y} + \dfrac{2y^2}{x^2+y^2} - \dfrac{y}{x+y}$

$$= \frac{2(x^2+y^2)}{x^2+y^2} - \frac{(x+y)}{x+y} = 2-1 = 1$$

Thus $\boldsymbol{x u_x + y u_y = 1}$

4. *If* $u = e^{ax-by}\sin(ax+by)$ *show that* $b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2abu$

>> $u = e^{ax-by}\sin(ax+by)$

$\therefore$ $\dfrac{\partial u}{\partial x} = e^{ax-by}\cos(ax+by)\cdot a + a\cdot e^{ax-by}\sin(ax+by)$

ie., $\dfrac{\partial u}{\partial x} = a\,e^{ax-by}\cos(ax+by) + au$ . . . (1)

Also $\dfrac{\partial u}{\partial y} = e^{ax-by}\cos(ax+by)\cdot b + (-b)\,e^{ax-by}\sin(ax+by)$

ie., $\dfrac{\partial u}{\partial y} = b\,e^{ax-by}\cos(ax+by) - bu$ . . . (2)

Now $b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y}$ by using (1) and (2) becomes

$= abe^{ax-by}\cos(ax+by) + abu - abe^{ax-by}\cos(ax+by) + abu$

$= 2abu$

Thus $\boldsymbol{b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2abu}$

5. *If* $u = e^{-c^2p^2t}(a\cos px + b\sin px)$, *show that* $u_t = c^2u_{xx}$

>> $u_t = e^{-c^2p^2t}(-c^2p^2)(a\cos px + b\sin px)$

ie., $u_t = -c^2p^2u$ . . . (1)

Also $u_x = e^{-c^2p^2t}(-ap\sin px + bp\cos px)$

$u_{xx} = e^{-c^2p^2t}(-ap^2\cos px - bp^2\sin px)$

ie., $u_{xx} = -p^2e^{-c^2p^2t}(a\cos px + b\sin px)$

or $\quad u_{xx} = -p^2 u$ by using the data.

$$\therefore \quad c^2 u_{xx} = -c^2 p^2 u \qquad \ldots (2)$$

Thus from (1) and (2) we have $u_t = c^2 u_{xx}$

$$>> \quad \frac{\partial u}{\partial r} = \frac{-2}{r^3} \cos 2\theta \qquad \ldots (1)$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{6}{r^4} \cos 2\theta \qquad \ldots (2)$$

Also $\dfrac{\partial u}{\partial \theta} = \dfrac{-2}{r^2} \sin 2\theta$ and $\dfrac{\partial^2 u}{\partial \theta^2} = -\dfrac{4}{r^2} \cos 2\theta$

$$\therefore \quad \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -\frac{4}{r^4} \cos 2\theta \qquad \ldots (3)$$

Thus from (1), (2) and (3) $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \theta^2}$

$$= \frac{6}{r^4} \cos 2\theta - \frac{2}{r^4} \cos 2\theta - \frac{4}{r^4} \cos 2\theta = 0$$

This proves the desired result.

$$>> \quad \frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 + (x^2/y^2)}} \cdot \frac{1}{y} = \frac{y}{\sqrt{y^2 + x^2}} \cdot \frac{1}{y} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 + (x^2/y^2)}} \cdot \frac{-x}{y^2} = \frac{y}{\sqrt{y^2 + x^2}} \cdot -\frac{x}{y^2} = \frac{-x}{y\sqrt{x^2 + y^2}}$$

Now $\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{x}{\sqrt{x^2 + y^2}} - \dfrac{x}{\sqrt{x^2 + y^2}} = 0$

This proves the desired result.

>> We have $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot -\frac{y}{x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

(a) Now $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1)-(-y)\,2y}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$ ... by quotient rule.

Thus $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (1)

Also $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$\therefore$ $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (2)

From (1) & (2) $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$

(b) $\dfrac{\partial^3 u}{\partial y^2\, \partial x} = \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)\right\}$

$$= \frac{\partial}{\partial y}\left\{\frac{y^2-x^2}{(x^2+y^2)^2}\right\}$$ ... by using (1)

$$= \frac{(x^2+y^2)^2 \cdot (2y) - (y^2-x^2)\,2\,(x^2+y^2)\,2y}{(x^2+y^2)^4} \quad \text{... by quotient rule}$$

$$= \frac{2y\,(x^2+y^2)\left[\,(x^2+y^2) - 2\,(y^2-x^2)\,\right]}{(x^2+y^2)^4}$$

Thus $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (3)

Also $\dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial}{\partial x}\,\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}\left\{\frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}\right\} = \frac{\partial}{\partial x}\left\{-\frac{2xy}{(x^2+y^2)^2}\right\}$$

$$= \frac{(x^2+y^2)^2\,(-2y) + 2xy\cdot 2\,(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{2y\,(x^2+y^2)\left[-(x^2+y^2)+4x^2\right]}{(x^2+y^2)^4}$$

$\therefore \quad \dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (4)

Also $\dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 u}{\partial x\,\partial y}\right)$

$$= \frac{\partial}{\partial y}\left\{\frac{y^2-x^2}{(x^2+y^2)^2}\right\} \quad \text{... by using (2)}$$

$$= \frac{(x^2+y^2)^2\cdot(2y) - (y^2-x^2)\,2\,(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y\,(x^2+y^2)\,[\,x^2+y^2-2y^2+2x^2\,]}{(x^2+y^2)^4}$$

Thus $\dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (5)

From (3), (4) and (5) the result follows.

9. [illegible]

>> We have $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

By data, $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$

$$\frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - \left\{ y^2 \cdot \frac{1}{1+(x/y)^2} \cdot -\frac{x}{y^2} + 2y \tan^{-1}(x/y) \right\}$$

(We have used the product rule while differentiating the second term)

ie., $$\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{x\,y^2}{x^2+y^2} - 2y \tan^{-1}(x/y)$$

ie., $$\frac{\partial z}{\partial y} = \frac{x\,(x^2+y^2)}{(x^2+y^2)} - 2y \tan^{-1}(x/y)$$

$\therefore$ $$\frac{\partial z}{\partial y} = x - 2y \tan^{-1}(x/y)$$

Now differentiating *w.r.t* x partially we have,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\left[\frac{1}{1+(x/y)^2} \cdot \frac{1}{y}\right]$$

or $$\frac{\partial^2 z}{\partial x\,\partial y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Thus $\boldsymbol{\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{x^2-y^2}{x^2+y^2}}$

10. [illegible]

>> $$u = \tan^{-1}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right] \text{ by data.}$$

$$\frac{\partial u}{\partial y} = \left(\frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\right) \cdot \frac{\partial}{\partial y}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2} \cdot x \left\{ \frac{\sqrt{1+x^2+y^2} \cdot 1 - \dfrac{y}{2\sqrt{1+x^2+y^2}} \times 2y}{(1+x^2+y^2)} \right\}$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)(1+y^2)} \cdot \left\{ \frac{1+x^2+y^2-y^2}{\sqrt{1+x^2+y^2}} \right\}$$

ie., $$\frac{\partial u}{\partial y} = \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

Now, $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{1+y^2}\,\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]$$

ie., $$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{1+y^2} \cdot \left\{ \frac{\sqrt{1+x^2+y^2} \cdot 1 - \dfrac{x \cdot 2x}{2\sqrt{1+x^2+y^2}}}{1+x^2+y^2} \right\}$$

$$= \frac{1}{1+y^2} \cdot \left\{ \frac{1+x^2+y^2-x^2}{(1+x^2+y^2)^{3/2}} \right\}$$

$$= \frac{1}{1+y^2} \cdot \frac{1+y^2}{(1+x^2+y^2)^{3/2}}$$

Thus $$\boldsymbol{\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}}$$

Verify that $u_{xy} = u_{yx}$ for the following functions:

[illegible]

11. $\boldsymbol{u = \sin^{-1}(y/x)}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{x}{\sqrt{x^2-y^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{1}{x} = \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{1}{x} = \frac{1}{\sqrt{x^2-y^2}}$$

Now $$\frac{\partial^2 u}{\partial x\,\partial y} = u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2-y^2}}\right)$$

ie., $$= \frac{\partial}{\partial x}\left\{(x^2-y^2)^{-1/2}\right\} = -\frac{1}{2}(x^2-y^2)^{-3/2} \cdot 2x$$

Thus $$u_{xy} = \frac{-x}{(x^2-y^2)^{3/2}} \qquad \ldots (1)$$

Also $\dfrac{\partial^2 u}{\partial y\,\partial x} = u_{yx} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{y}{x\sqrt{x^2-y^2}}\right)$

ie., $$= -\frac{1}{x}\left\{\frac{\sqrt{x^2-y^2}\cdot 1 - y\cdot\dfrac{-2y}{2\sqrt{x^2-y^2}}}{x^2-y^2}\right\}$$

$$= -\frac{1}{x}\left\{\frac{x^2-y^2+y^2}{(x^2-y^2)^{3/2}}\right\} = -\frac{x}{(x^2-y^2)^{3/2}}$$

Thus $u_{yx} = \dfrac{-x}{(x^2-y^2)^{3/2}}$

From (1) and (2) $u_{xy} = u_{yx}$. **The result is verified.**

12. $u = x^y$

$\therefore$ $\dfrac{\partial u}{\partial x} = y\,x^{y-1}$ ... *Treating* y *as a constant we have used* $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$

$\dfrac{\partial u}{\partial y} = x^y \log x$ ... *Treating* x *as a constant we have used* $\dfrac{d}{dx}(a^x) = a^x \log a$

$u_{xy} = \dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}(x^y \log x)$

ie., $= x^y \cdot \dfrac{1}{x} + y\,x^{y-1}\cdot \log x$ ... *(by product rule)*

$= x^{y-1} + y\,x^{y-1}\log x$

Thus $u_{xy} = x^{y-1}(1 + y\log x)$ ...(1)

Also $u_{yx} = \dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}(y\,x^{y-1})$

ie., $= y\cdot x^{y-1}\log x + x^{y-1}\cdot 1 = x^{y-1}(y\log x + 1)$

Thus $u_{yx} = x^{y-1}(1 + y\log x)$ ...(2)

From (1) and (2) $u_{xy} = u_{yx}$. **The result is verified.**

$$\frac{\partial u}{\partial x} = e^x \cdot \cos y + e^x (x \cos y - y \sin y) \quad \text{...}\textit{by product rule.}$$

$$\frac{\partial u}{\partial y} = e^x(-x \sin y - y \cos y - \sin y) = -e^x (x \sin y + y \cos y + \sin y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left\{e^x (x \sin y + y \cos y + \sin y)\right\}$$

ie., $= -\left\{e^x \cdot \sin y + e^x (x \sin y + y \cos y + \sin y)\right\}$

Thus $u_{xy} = -e^x (2 \sin y + x \sin y + y \cos y)$ ...(1)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left\{e^x (\cos y + x \cos y - y \sin y)\right\}$$

ie., $= e^x(-\sin y - x \sin y - y \cos y - \sin y)$

Thus $u_{yx} = -e^x (2 \sin y + x \sin y + y \cos y)$ ...(2)

From (1) and (2), $u_{xy} = u_{yx}$. **The result is verified.**

>> We have $z = \tan(y + ax) + (y - ax)^{3/2}$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax) \cdot a + \frac{3}{2}(y - ax)^{1/2} \cdot (-a)$$

Differentiating *w.r.t* x again partially,

$$\frac{\partial^2 z}{\partial x^2} = 2\sec(y + ax) \cdot \sec(y + ax)\tan(y + ax) \cdot a^2 + \frac{3}{2} \cdot \frac{1}{2}(y - ax)^{-1/2} \cdot a^2$$

$$\frac{\partial^2 z}{\partial x^2} = a^2\left\{2\sec^2(y + ax)\tan(y + ax) + \frac{3}{4}(y - ax)^{-1/2}\right\} \quad \text{...(1)}$$

Now, $\dfrac{\partial z}{\partial y} = \sec^2(y + ax) + \dfrac{3}{2}(y - ax)^{1/2}$

Also $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = 2\sec(y + ax) \cdot \sec(y + ax) \cdot \tan(y + ax) + \dfrac{3}{2} \cdot \dfrac{1}{2}(y - ax)^{-1/2}$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2} \quad \ldots (2)$$

Using (2) in (1) we have $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$

Thus $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$

15. [illegible]

>> $v = e^{a\theta}\cos(a\log r)$

$$\frac{\partial v}{\partial r} = -e^{a\theta} \cdot \sin(a\log r) \cdot \frac{a}{r} \quad \ldots (1)$$

$$\therefore \quad \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\right) = -e^{a\theta}\left\{\sin(a\log r)\cdot\left(-\frac{a}{r^2}\right) + \cos(a\log r)\cdot\frac{a^2}{r^2}\right\}$$

ie., $$\frac{\partial^2 v}{\partial r^2} = \frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r) \quad \ldots (2)$$

Next, $\dfrac{\partial v}{\partial \theta} = ae^{a\theta}\cos(a\log r)$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial^2 v}{\partial \theta^2} = a^2 e^{a\theta}\cos(a\log r) \quad \ldots (3)$$

Consider $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2}$ by using (1), (2) and (3).

$$= \left\{\frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\}$$

$$+ \left\{-\frac{a}{r^2}e^{a\theta}\sin(a\log r)\right\} + \left\{\frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\} = 0$$

Thus $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0.$

16. *If* $\theta = t^n e^{-r^2/4t}$ *find the value of* n *such that* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

>> We have $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial t} = t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right) + n\,t^{n-1} e^{-r^2/4t} \quad \text{...by product rule.}$$

ie., $$\frac{\partial\theta}{\partial t} = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} + n\,t^{n-1} e^{-r^2/4t} \qquad \ldots (1)$$

Next $$\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t} \cdot \left(-\frac{2r}{4t}\right) = -\frac{1}{2} t^{n-1}\, r\, e^{-r^2/4t}$$

$$\therefore \quad r^2\frac{\partial\theta}{\partial r} = -\frac{1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

Now $$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{t^{n-1}}{2}\left\{r^3 e^{-r^2/4t} \cdot \left(-\frac{2r}{4t}\right) + 3r^2 e^{-r^2/4t}\right\}$$

$$= \frac{1}{4} r^4 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} r^2 e^{-r^2/4t}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} e^{-r^2/4t} \qquad \ldots (2)$$

Using (1) & (2) in the equation, $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$ we have,

$$\frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} e^{-r^2/4t} = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} + n\,t^{n-1} e^{-r^2/4t}$$

or $$-\frac{3}{2} t^{n-1} e^{-r^2/4t} = n\,t^{n-1} e^{-r^2/4t} \Rightarrow -\frac{3}{2} = n$$

Thus the required value of $n = -3/2$.

[illegible]

>> $u = e^{ax+by} f(ax-by)$, by data.

$$\frac{\partial u}{\partial x} = e^{ax+by} \cdot f'(ax-by)\,a + a\,e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial x} = a\,e^{ax+by} f'(ax-by) + a\,u \qquad \ldots (1)$$

Next, $$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by) \cdot (-b) + b\,e^{ax+by} f(ax-by)$$

or $\dfrac{\partial u}{\partial y} = -b\,e^{ax+by} f'(ax-by) + bu$ ... (2)

Now consider L.H.S $= b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y}$ by using (1) and (2).

$$= b\left\{a e^{ax+by} f'(ax-by) + au\right\} + a\left\{-b e^{ax+by} f'(ax-by) + bu\right\}$$

$$= ab e^{ax+by} f'(ax-by) + abu - ab e^{ax+by} f'(ax-by) + abu$$

$$= 2abu = \text{R.H.S}$$

Thus $\boldsymbol{b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2\,a\,b\,u.}$

>> We have $u = f(x+ct) + g(x-ct)$

$$u_x = \frac{\partial u}{\partial x} = f'(x+ct)\cdot 1 + g'(x-ct)\cdot 1$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x-ct) \qquad \ldots (1)$$

$$u_t = \frac{\partial u}{\partial t} = f'(x+ct)\cdot(c) + g'(x-ct)\cdot(-c)$$

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = f''(x+ct)\cdot(c^2) + g''(x-ct)(c^2)$$

ie., $u_{tt} = \dfrac{\partial^2 u}{\partial t^2} = c^2\left[f''(x+ct) + g''(x-ct\right]$ ... (2)

Using (1) in (2) we have $\boldsymbol{u_{tt} = c^2 u_{xx}}$

>> We have $u = \dfrac{1}{r}\left[f(r-at) + \phi(r+at)\right]$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{1}{r}\left[f'(r-at)\cdot(-a) + \phi'(r+at)\cdot a\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{1}{r}\left[f''(r-at)\,a^2 + \phi''(r+at)\,a^2\right]$$

ie., $$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r}\left[f''(r-at)+\phi''(r+at)\right] \qquad \ldots(1)$$

Next $$\frac{\partial u}{\partial r} = \frac{1}{r}\left[f'(r-at)\cdot 1+\phi'(r+at)\cdot 1\right]+\left[f(r-at)+\phi(r+at)\right]\cdot\left(-\frac{1}{r^2}\right)$$

$$\therefore \quad r^2\frac{\partial u}{\partial r} = r\left[f'(r-at)+\phi'(r+at)\right]-\left[f(r-at)+\phi(r+at)\right]$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r\left[f''(r-at)+\phi''(r+at)\right]+\left[f'(r-at)+\phi'(r+at)\right]$$

$$-\left[f'(r-at)+\phi'(r+at)\right]$$

$$= r\left[f''(r-at)+\phi''(r+at)\right]$$

$$\therefore \quad \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{a^2}{r}\left[f''(r-at)+\phi''(r+at)\right] \qquad \ldots(2)$$

From (1) & (2) $$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

>> $$z = f_1(y-3x)+f_2(y+2x)+\sin x - y\cos x$$

$$\frac{\partial z}{\partial x} = f_1'(y-3x)\cdot(-3)+f_2'(y+2x)\cdot 2+\cos x+y\sin x$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_1''(y-3x)\cdot 9+f_2''(y+2x)\cdot 4-\sin x+y\cos x$$

ie., $$\frac{\partial^2 z}{\partial x^2} = 9f_1''(y-3x)+4f_2''(y+2x)-\sin x+y\cos x \qquad \ldots(1)$$

$$\frac{\partial z}{\partial y} = f_1'(y-3x)+f_2'(y+2x)-\cos x$$

$$\therefore \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_1''(y-3x)+f_2''(y+2x) \qquad \ldots(2)$$

Also $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_1''(y-3x)\cdot(-3)+f_2''(y+2x)\cdot 2+\sin x$$

ie., $\dfrac{\partial^2 z}{\partial x\,\partial y} = -3f_1''(y-3x) + 2f_2''(y+2x) + \sin x$ ... (3)

Consider $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\,\partial y} - 6\dfrac{\partial^2 z}{\partial y^2}$ using (1), (2) and (3).

$$= \left[9f_1''(y-3x) + 4f_2''(y+2x) - \sin x + y\cos x\right]$$

$$+\left[-3f_1''(y-3x) + 2f_2''(y+2x) + \sin x\right] - 6\left[f_1''(y-3x) + f_2''(y+2x)\right]$$

$$= 9f_1''(y-3x) - 9f_1''(y-3x) + 6f_2''(y+2x) - 6f_2''(y+2x) + y\cos x$$

$$= y\cos x$$

Thus $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\,\partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

>> By data $u = \log\sqrt{x^2+y^2+z^2} = \dfrac{1}{2}\log(x^2+y^2+z^2)$

The given u is a symmetric function of $x, y, z,$

(It is enough if we compute only one of the required partial derivative)

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

ie., $$= \frac{(x^2+y^2+z^2)\,1 - x\cdot 2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$ ... (1)

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$ ... (2)

$\dfrac{\partial^2 u}{\partial z^2} = \dfrac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$ ... (3)

Adding (1), (2), and (3) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}=\frac{1}{x^2+y^2+z^2}$$

Thus $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$

>> $u=\log(\tan x+\tan y+\tan z)$ is a symmetric function.

$$u_x=\frac{\sec^2 x}{\tan x+\tan y+\tan z}$$

$\therefore$ $$\sin 2x\,u_x=\frac{(2\sin x\cos x)\sec^2 x}{\tan x+\tan y+\tan z}$$

or $$\sin 2x\,u_x=\frac{2\tan x}{\tan x+\tan y+\tan z}\qquad\ldots(1)$$

Similarly $$\sin 2y\,u_y=\frac{2\tan y}{\tan x+\tan y+\tan z}\qquad\ldots(2)$$

$$\sin 2z\,u_z=\frac{2\tan z}{\tan x+\tan y+\tan z}\qquad\ldots(3)$$

Adding (1),(2) and (3) we get,

$$\sin 2x\,u_x+\sin 2y\,u_y+\sin 2z\,u_z=\frac{2(\tan x+\tan y+\tan z)}{(\tan x+\tan y+\tan z)}=2$$

Thus $\mathbf{\sin 2x\,u_x+\sin 2y\,u_y+\sin 2z\,u_z=2}$

>> $u=(x^2+y^2+z^2)^{-1/2}$ is a symmetric function of $x,\ y,\ z,$

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x=-(x^2+y^2+z^2)^{-3/2}\cdot x$$

$\therefore$ $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=-\frac{\partial}{\partial x}\left\{(x^2+y^2+z^2)^{-3/2}\cdot x\right\}$$

$$=-\left\{(x^2+y^2+z^2)^{-3/2}\cdot 1+x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x\right\}$$

$$=-\left\{(x^2+y^2+z^2)^{-3/2}-3x^2(x^2+y^2+z^2)^{-5/2}\right\}$$

ie., $$\frac{\partial^2 u}{\partial x^2} = 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \qquad \ldots (1)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = 3y^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \qquad \ldots (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \qquad \ldots (3)$$

Adding the results (1), (2) and (3) we have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= 3 (x^2 + y^2 + z^2)^{-5/2} (x^2 + y^2 + z^2) - 3 (x^2 + y^2 + z^2)^{-3/2}$$

$$= 3 (x^2 + y^2 + z^2)^{-3/2} - 3 (x^2 + y^2 + z^2)^{-3/2} = 0$$

Thus $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

>> $u = \log (x^3 + y^3 + z^3 - 3xyz)$ is a symmetric function.

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \qquad \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \qquad \ldots (3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3 (x^2 + y^2 + z^2 - xy - yz - zx)}{(x^3 + y^3 + z^3 - 3xyz)}$$

Recalling a standard elementary result,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)$$

we have,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\,(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)}$$

Thus $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\dfrac{3}{x+y+z}$

Further $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right),$$ by using the earlier result.

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}=\frac{-9}{(x+y+z)^2}$$

Thus $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=\dfrac{-9}{(x+y+z)^2}$

>> $z=\dfrac{x^2+y^2}{x+y}$ by data, is a symmetric function of $x,\ y$

$$\frac{\partial z}{\partial x}=\frac{(x+y)\,2x-(x^2+y^2)\,1}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2} \qquad \ldots(1)$$

Similarly $\dfrac{\partial z}{\partial y}=\dfrac{y^2+2xy-x^2}{(x+y)^2}$ ... (2)

Now $\left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]=\frac{(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}$

$$=\frac{2(x^2-y^2)}{(x+y)^2}=\frac{2(x-y)(x+y)}{(x+y)^2}=\frac{2(x-y)}{(x+y)}$$

$$\therefore \quad \left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]^2=4\frac{(x-y)^2}{(x+y)^2} \qquad \ldots(3)$$

Now, $4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]$

$$=4\left[\frac{(x+y)^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

ie., $$4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]=4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]=4\frac{(x-y)^2}{(x+y)^2} \qquad \ldots(4)$$

Thus from (3) and (4) $\left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]^2=4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]$

This is analogous to the concept of the differentiation of implict functions in the case of ordinary derivatives.

Given a relation of the form $f(x, y, z)=c$, we need to identify the dependent variable and the associated independent variables based on the required partial derivatives. For example if $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x\,\partial y}, \frac{\partial^2 z}{\partial y^2}$ etc, are required we have to obviously infer that z is a function of x and y.

Suppose we have $x=f(u, v)$ and $y=g(u, v)$ and let us suppose that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ is required. In this case we have to express u in terms of x and y as a single equation by eliminating v from the given relations.

It is important to note the following difference

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \text{ and vice - versa, where as}$$

$$\frac{\partial u}{\partial x} \neq \frac{1}{\left(\frac{\partial x}{\partial u}\right)} \text{ and } \frac{\partial u}{\partial y} \neq \frac{1}{\left(\frac{\partial y}{\partial u}\right)} \text{ etc.}$$

WORKED PROBLEMS

26. *If* $x^x y^y z^z = c$, *show that* $\dfrac{\partial^2 z}{\partial x\, \partial y} = -\dfrac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$

and hence deduce that $\dfrac{\partial^2 z}{\partial x\, \partial y} = -(x \log ex)^{-1}$ *when* $x = y = z$

>> We have $x^x y^y z^z = c$ and we have to treat z as a function of x and y in order to find the required partial derivative.

Taking logarithms on both sides of the given equation we get

$$x \log x + y \log y + z \log z = \log c$$

Differentiating partially *w.r.t.* y bearing in mind that z is a function of x & y we get

$$0 + \left(y \cdot \frac{1}{y} + 1 \cdot \log y\right) + \left(z \cdot \frac{1}{z} + 1 \cdot \log z\right)\frac{\partial z}{\partial y} = 0$$

$$\therefore \quad \frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$$

Now $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = -\dfrac{\partial}{\partial x}\left\{\dfrac{(1+\log y)}{(1+\log z)}\right\}$

ie., $$\frac{\partial^2 z}{\partial x\, \partial y} = -\left\{\frac{(1+\log z)\cdot 0 - (1+\log y)\cdot \frac{1}{z}\frac{\partial z}{\partial x}}{(1+\log z)^2}\right\} \quad \ldots (1)$$

Taking a note that the given function is symmetric we can write,

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \quad \ldots (2)$$

Using (2) in (1)we get,

$$\boldsymbol{\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}}$$

Further when $x = y = z$, the R.H.S of the above expression on replacing y and z by x assumes the form

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x(\log e+\log x)} = -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

Thus $\left[\frac{\partial^2 z}{\partial x\,\partial y}\right]_{x=y=z} = -(x\log ex)^{-1}$

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27. If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

>> $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ is a symmetric function.

(We do not prefer to write the expression for r)

Differentiating partially *w.r.t* x on both sides, we have,

$$2r\frac{\partial r}{\partial x} = 2(x-a) \text{ or } \frac{\partial r}{\partial x} = \frac{(x-a)}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{(x-a)}{r}\right]$$

Applying quotient rule we get,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r\cdot 1-(x-a)\cdot\frac{\partial r}{\partial x}}{r^2} = \frac{r-[(x-a)\cdot(x-a)/r]}{r^2}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} = \frac{r^2-(x-a)^2}{r^3} \qquad \ldots(1)$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{r^2-(y-b)^2}{r^3}$ ...(2)

$$\frac{\partial^2 r}{\partial z^2} = \frac{r^2-(z-c)^2}{r^3} \qquad \ldots(3)$$

Adding the results (1), (2) and (3) we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{1}{r^3}\left\{3r^2-[(x-a)^2+(y-b)^2+(z-c)^2]\right\}$$

$$= \frac{1}{r^3}(3r^2-r^2) = \frac{2r^2}{r^3} = \frac{2}{r}$$

Thus $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

---

28. [illegible] $x = r\cos\theta$ [illegible]

(a) [illegible] (b) [illegible]

>> Observing the desired result, we need to first express r as a function of x & y.

$$x = r\cos\theta,\ y = r\sin\theta \text{ gives } x^2+y^2 = r^2$$

Now consider $r^2 = x^2+y^2$ which is a symmetric function.

Differentiating partialy *w.r.t* x on both sides we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial r}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{x}{r}\right)$

Applying quotient rule, we have,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r\cdot 1 - x\cdot\dfrac{\partial r}{\partial x}}{r^2} = \frac{r - x\cdot\dfrac{x}{r}}{r^2} = \frac{r^2-x^2}{r^3}$$

Similarly $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{r^2-y^2}{r^3}$

$$\text{L.H.S} = \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3} = \frac{1}{r^3}\left[2r^2-(x^2+y^2)\right] = \frac{2r^2-r^2}{r^3} = \frac{1}{r}$$

$$\text{Also R.H.S} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\} = \frac{1}{r}\left\{\frac{x^2}{r^2}+\frac{y^2}{r^2}\right\}$$

ie., $\quad = \dfrac{1}{r}\left\{\dfrac{x^2+y^2}{r^2}\right\} = \dfrac{1}{r}\cdot\dfrac{r^2}{r^2} = \dfrac{1}{r}$

Thus **L.H.S = R.H.S**

(b) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \dfrac{r\cos\theta}{r} = \cos\theta$

Since $x = r\cos\theta$, $\dfrac{\partial x}{\partial r} = \cos\theta$

$\therefore \quad \dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$

29 *If* $x = r\cos\theta$, $y = r\sin\theta$ *show that*

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2}{\partial x^2}(\log r) = -\frac{\partial^2}{\partial y^2}(\log r) = \frac{-\cos 2\theta}{r^2}$$

>> **Observing the required partial derivatives, we need to express r and θ in terms of x, y.**

$$x = r\cos\theta,\quad y = r\sin\theta \text{ gives } x^2+y^2 = r^2$$

Also $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$ or $\tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}(y/x)$

(i) Consider $\theta = \tan^{-1}(y/x)$

$$\therefore \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

Thus $\dfrac{\partial^2\theta}{\partial x\,\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (1)

(ii) Let $u = \log r = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

Now $\dfrac{\partial u}{\partial x} = \dfrac{1}{2}\cdot\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (2)

Since $u = \log\sqrt{x^2+y^2}$ is a symmetric function, we have similarly

$\dfrac{\partial^2 u}{\partial y^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2} = \dfrac{-(y^2-x^2)}{(x^2+y^2)^2}$ ... (3)

Thus from (1), (2) and (3) we have

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2} \quad \text{where } u = \log r.$$

Since each of these is equal to $\dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ we have on substituting $x = r\cos\theta$, $y = r\sin\theta$ the same becomes

$$\frac{r^2(\sin^2\theta - \cos^2\theta)}{(r^2)^2} = \frac{-(\cos^2\theta - \sin^2\theta)}{r^2} = \frac{-\cos 2\theta}{r^2}$$

Thus $\dfrac{\partial^2\theta}{\partial x\,\partial y} = \dfrac{\partial^2}{\partial x^2}(\log r) = -\dfrac{\partial^2}{\partial y^2}(\log r) = -\dfrac{\cos 2\theta}{r^2}$

---

30. *If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, then show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r).$$ *Hence deduce the expression if $u = r^n$*

\>> $u = f(r)$ where r is a function of x, y, z.

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} \qquad \ldots (1)$$

But $r = \sqrt{x^2 + y^2 + z^2}$ gives $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$

Thus $\dfrac{\partial u}{\partial x} = f'(r) \cdot \dfrac{x}{r}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left\{f'(r) \cdot \frac{x}{r}\right\}.$$

Applying product rule we have,

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r \cdot 1 - x \cdot \dfrac{\partial r}{\partial x}}{r^2}\right] + f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\left[r - x \cdot \frac{x}{r}\right] + f''(r)\frac{x}{r} \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r^2} \cdot \frac{(r^2 - x^2)}{r} + f''(r) \cdot \frac{x^2}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3}(r^2 - x^2) + f''(r) \cdot \frac{x^2}{r^2} \qquad \ldots (2)$$

$$\text{Similarly} \quad \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}(r^2 - y^2) + f''(r) \cdot \frac{y^2}{r^2} \qquad \ldots (3)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{f'(r)}{r^3}(r^2 - z^2) + f''(r) \cdot \frac{z^2}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{f'(r)}{r^3}\left\{(r^2 - x^2) + (r^2 - y^2) + (r^2 - z^2)\right\} + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{f'(r)}{r^3}\left\{3r^2 - (x^2 + y^2 + z^2)\right\} + \frac{f''(r)}{r^2} \cdot r^2 \qquad \because x^2 + y^2 + z^2 = r^2$$

$$= \frac{f'(r)}{r^3} \cdot 2r^2 + f''(r) = \frac{2}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$

Next if $u = r^n$ then $f'(r) = nr^{n-1}$, $f''(r) = n(n-1)r^{n-2}$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= n(n-1)r^{n-2} + \frac{2}{r} \cdot n r^{n-1} = n(n-1)r^{n-2} + 2n r^{n-2} = r^{n-2}(n^2 - n + 2n)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = (n^2 + n) r^{n-2} = n(n+1) r^{n-2}$

---

31. *If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

>> Observing the required partial derivatives we conclude that u must be a function of x, y. But $u = f(r)$ by data and hence we need to have r as a function of x, y. Since $x = r\cos\theta$, $y = r\sin\theta$ we have $x^2 + y^2 = r^2$.

$\therefore$ we have $u = f(r)$ where $r = \sqrt{x^2 + y^2}$.

This example is virtually similar to the previous example wherein we had another term z^2. Proceeding on the same lines one can easily obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3}(r^2 - x^2) + \frac{f''(r)}{r^2} \cdot x^2 \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}(r^2 - y^2) + \frac{f''(r)}{r^2} \cdot y^2$$

Adding these results we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left\{2r^2 - (x^2 + y^2)\right\} + \frac{f''(r)}{r^2}(x^2 + y^2)$$

$$= \frac{f'(r)}{r^3} \cdot r^2 + \frac{f''(r)}{r^2} \cdot r^2 = \frac{1}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

---

32. If $r^2 = x^2 + y^2 + z^2$, then $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right) e^r = \left(1 + \dfrac{2}{r}\right) e^r$

>> If $u = e^r$, then we have to show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \left(1 + \dfrac{2}{r}\right) e^r$

Here $u = e^r$ where $r^2 = x^2 + y^2 + z^2$, is a symmetric function.

$$\therefore \quad \frac{\partial u}{\partial x} = e^r \frac{\partial r}{\partial x}$$

But $r^2 = x^2 + y^2 + z^2$ and hence $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Thus $\dfrac{\partial u}{\partial x} = e^r \cdot \dfrac{x}{r}$

Now $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(e^r \cdot \dfrac{x}{r}\right)$

ie., $$= e^r \left[\frac{r \cdot 1 - x\dfrac{\partial r}{\partial x}}{r^2}\right] + e^r \frac{\partial r}{\partial x} \cdot \frac{x}{r}$$

$$= \frac{e^r}{r^2}\left[r - x \cdot \frac{x}{r}\right] + e^r \cdot \frac{x}{r} \cdot \frac{x}{r}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{e^r}{r^3}(r^2 - x^2) + \frac{e^r}{r^2} \cdot x^2 \qquad \ldots (1)$$

Similarly $\quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{e^r}{r^3}(r^2 - y^2) + \dfrac{e^r}{r^2} \cdot y^2$ . . . (2)

$\dfrac{\partial^2 u}{\partial z^2} = \dfrac{e^r}{r^3}(r^2 - z^2) + \dfrac{e^r}{r^2} \cdot z^2$ . . . (3)

Adding (1), (2) and (3) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{e^r}{r^3}\left[3r^2 - (x^2 + y^2 + z^2)\right] + \frac{e^r}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{e^r}{r^3} \cdot 2r^2 + \frac{e^r}{r^2} \cdot r^2 = e^r \cdot \frac{2}{r} + e^r$$

Thus $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = e^r\left(\dfrac{2}{r} + 1\right)$

---

33. *If* $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$, *show that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

\>> We need to treat u as a function of x, y, z.

Consider $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$

Differentiating partially *w.r.t* x we get

$$\left\{x^2 \cdot -\frac{1}{(a^2 + u)^2} u_x + 2x \cdot \frac{1}{a^2 + u}\right\} + \left\{y^2 \cdot -\frac{1}{(b^2 + u)^2} u_x\right\}$$

$$+ \left\{z^2 \cdot -\frac{1}{(c^2 + u)^2} u_x\right\} = 0$$

or $\quad u_x\left[\dfrac{x^2}{(a^2 + u)^2} + \dfrac{y^2}{(b^2 + u)^2} + \dfrac{z^2}{(c^2 + u)^2}\right] = \dfrac{2x}{a^2 + u}$

We get similar equations related to u_y and u_z with the repetition of the expression in the square bracket [ ].

Let $\quad v = \dfrac{x^2}{(a^2 + u)^2} + \dfrac{y^2}{(b^2 + u)^2} + \dfrac{z^2}{(c^2 + u)^2}$ for convenience.

Now we have three equations :

$$u_x \cdot v = \frac{2x}{a^2 + u}, \quad u_y \cdot v = \frac{2y}{b^2 + u}, \quad u_z \cdot v = \frac{2z}{c^2 + u} \quad \ldots (1)$$

Squaring and adding these we get,

$$(u_x^2+u_y^2+u_z^2)\,v^2 = 4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] = 4v$$

$$\therefore \quad \text{L.H.S} = u_x^2+u_y^2+u_z^2 = \frac{4v}{v^2} = \frac{4}{v} \qquad \ldots (2)$$

Taking a note of the required expression in R.H.S, let us multiply the three equations present in (1) respectively by x, y, z and add.

$$\therefore \quad x\,u_x\,v + y\,u_y\,v + z\,u_z\,v = 2\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

or
$$2\,(x\,u_x + y\,u_y + z\,u_z) = \frac{4}{v}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

But $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$, by data.

Hence
$$2\,(x\,u_x + y\,u_y + z\,u_z) = \frac{4}{v} \qquad \ldots (3)$$

$\therefore$ from (2) and (3), $u_x^2+u_y^2+u_z^2 = 2\,(x\,u_x + y\,u_y + z\,u_z)$

[illegible] Total differentiation

If $u = f(x, y)$ then the total differential or the exact differential of u is defined as

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad \ldots (1)$$

Differentiation of composite and implicit functions

If $u = f(x, y)$ where x and y are functions of the independent variable t then u *is said to be a composite function of the single variable* t.

Also if $u = f(x, y)$ where both x and y are functions of two independent variables r, s then u *is said to be a composite function of the two variables* r *and* s.

The principle of differentiation of composite function is very much similar to that of the function of a function rule associated with the ordinary derivative of a function of a single independent variable.

We discuss two types involving partial derivatives.

Type - 1 : Total derivative

If $u = f(x, y)$ where $x = x(t)$ and $y = y(t)$ then u is a composite function of the single variable t. Therefore in principle we should be able to differentiate u with respect to t which is an ordinary derivative.

Thus we have with reference to (1),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \ldots (2)$$

This is called as the total derivative of u.

Type - (ii). Chain rule

If $u = f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$ then u is a composite function of two independent variables r, s. Therefore in principle we should we able to differentiate u *w.r.t* r and also *w.r.t* s partially. Thus we have the following chain rules for the two partial derivatives. It is convenient to write the rule having the data analysed in the following format.

$$u \to (x, y) \to (r, s) \;\Rightarrow\; u \to (r, s) \left\langle \begin{array}{l} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial s} \end{array} \right.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \;;\; \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \qquad \ldots (3)$$

Note - 1. *The rules (2) and (3) can be established from the basic limit form definition of a partial derivative.*

2 . *The rules (2) and (3) can be extended to functions involving more than two independent variables.*

3 . *The rules (2) and (3) can be successively applied for getting higher order derivatives of the given function.*

4. *The symbol $\to$ is used only to indicate the composition of the variables so that the associated rule can be written conveniently.*

Corollary : Differentiation of implicit functions

Let $u = f(x, y)$ and let y be a function of x and also $f(x, y) = c$, c being a constant.

ie., $u = f(x, y)$ where $y = y(x)$. Hence by the rule of the total derivative,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

ie., $$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Since $u = f(x, y) = c$ we have $\dfrac{du}{dx} = 0$ and the above equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = -\frac{u_x}{u_y}$$

Thus we can say that if $u = f(x, y) = c$ then

$$\frac{dy}{dx} = -\frac{u_x}{u_y} \quad \text{or} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Remark : *It is a known fact that a function $f(x, y) = c$ is called an implicit function and we are conversant in finding the derivative of an implict function. Here we have a formula for $\frac{dy}{dx}$ in terms of partial derivatives and the same can be successively applied for higher order derivatives.*

WORKED PROBLEMS

Working procedure for problems

- We have to analyse the composition of the variables and write the appropriate formula.
- We then substitute for the possible derivatives in the formula and simplify according to the requirement of the desired result.
- Sometimes we may have to change the composition of the variables so as to achieve the desired result.

Find the total differential of the following functions.

34. $u = x^3 + xy^2 + x^2y + y^3$ 35. $x = r\sin\theta\cos\phi$

36. [illegible]   37. [illegible]

34. We have $u = x^3 + xy^2 + x^2y + y^3$; $u \to (x, y)$

$$\therefore \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus $du = (3x^2 + y^2 + 2xy)\,dx + (2xy + x^2 + 3y^2)\,dy$

---

35. We have $x = r\sin\theta\cos\phi$; $x \to (r, \theta, \phi)$

$$\therefore \quad dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

Thus $dx = (\sin\theta\cos\phi)\,dr + (r\cos\theta\cos\phi)\,d\theta + (-r\sin\theta\sin\phi)\,d\phi$

---

36. **We have** $\psi = u_1 u_2^2 + u_2 u_3^2 + u_3 u_1^2 \; ; \; \psi \to (u_1, u_2, u_3)$

$$\therefore \quad d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3$$

Thus $d\psi = (u_2^2 + 2u_1 u_3)\, du_1 + (u_3^2 + 2u_1 u_2)\, du_2 + (u_1^2 + 2u_2 u_3)\, du_3$

---

37. **We have** $\phi = xy^2 z^3 \; ; \; \phi \to (x, y, z)$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

Thus $d\phi = (y^2 z^3)\, dx + (2xyz^3)\, dy + (3xy^2 z^2)\, dz$

Find the total derivative of the following functions and also verify the result by direct substitution

38. [illegible]

39. [illegible]

40. [illegible]

38. $z = xy^2 + x^2 y \; ; \; x = at, \; y = 2at$

$\{z \to (x, y) \to t\} \Rightarrow z \to t$ & $\frac{dz}{dt}$ is the total derivative.

$$\therefore \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)\, a + (2xy + x^2)\, 2a$$

$$= (4a^2 t^2 + 4a^2 t^2)\, a + (4a^2 t^2 + a^2 t^2)\, 2a$$

$$= (8a^2 t^2)\, a + (5a^2 t^2)\, 2a = 8a^3 t^2 + 10a^3 t^2 = 18a^3 t^2$$

$\therefore$ the total derivative $\frac{dz}{dt} = 18a^3 t^2$ . . . (1)

Now by direct substitution we have,

$$z = xy^2 + x^2 y = (at)(2at)^2 + (at)^2 (2at) = 4a^3 t^3 + 2a^3 t^3$$

ie., $z = 6a^3 t^3$

Now differentiating *w.r.t* t,

$$\frac{dz}{dt} = 6a^3 \cdot 3t^2 = 18a^3 t^2$$

From (1) & (2) the **result is verified.**

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39. $u = x^2 + y^2 - z^2,\ x = e^t,\ y = e^t \cosh t,\ z = e^t \sinh t$

$\{ u \to (x, y, z) \to t \} \Rightarrow u \to t$ & $\dfrac{du}{dt}$ is the total derivative.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2x)(e^t) + (2y)(e^t \sinh t + e^t \cosh t)$$
$$- (2z)(e^t \cosh t + e^t \sinh t)$$

Substituting for x, y, z, we get

$$= 2e^t \left\{ e^t + e^t \cosh t (\sinh t + \cosh t) - e^t \sinh t (\cosh t + \sinh t) \right\}$$
$$= 2e^t \cdot e^t \left\{ 1 + \cosh t \sinh t + \cosh^2 t - \sinh t \cosh t - \sinh^2 t \right\}$$
$$= 2e^{2t} \left\{ 1 + (\cosh^2 t - \sinh^2 t) \right\}$$
$$= 2e^{2t}(1 + 1) = 4e^{2t}$$

$\therefore$ the total derivative $\dfrac{du}{dt} = 4e^{2t}$ ... (1)

Now by direct substitution we have,

$$u = x^2 + y^2 - z^2 = (e^t)^2 + (e^t \cosh t)^2 - (e^t \sinh t)^2$$
$$= e^{2t}\left\{ 1 + (\cosh^2 t - \sinh^2 t) \right\} = e^{2t}(1 + 1) = 2e^{2t}$$

ie., $u = 2e^{2t}$

Differentiating *w.r.t* t we get,

$\dfrac{du}{dt} = 2(2e^{2t}) = 4e^{2t}$ ... (2)

From (1) and (2) the result is verified.

---

40. $u = xy + yz + zx\ ;\ x = t \cos t,\ y = t \sin t,\ z = t$

$\{ u \to (x, y, z) \to t \} \Rightarrow u \to t$ & $\dfrac{du}{dt}$ is the total derivative.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y + z)(-t \sin t + \cos t) + (x + z)(t \cos t + \sin t) + (y + x)(1)$$
$$= (t \sin t + t)(-t \sin t + \cos t) + (t \cos t + t)(t \cos t + \sin t)$$
$$+ (t \sin t + t \cos t)$$

$$= (-t^2\sin^2 t - t^2\sin t + t\sin t\cos t + t\cos t)$$
$$+ (t^2\cos^2 t + t^2\cos t + t\sin t\cos t + t\sin t) + (t\sin t + t\cos t)$$

$$= t^2(\cos^2 t - \sin^2 t) + t^2(\cos t - \sin t)$$
$$+ 2t\sin t\cos t + 2t(\cos t + \sin t)$$

Now at $t = \pi/4$ we have $\cos t = \sin t = 1/\sqrt{2}$

$$\therefore \quad \left(\frac{du}{dt}\right)_{t=\pi/4} = 0 + 0 + 2\cdot\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 2\cdot\frac{\pi}{4}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

ie.,
$$\left(\frac{du}{dt}\right)_{t=\pi/4} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}} \qquad \ldots(1)$$

Now by direct substitution we have,

$$u = xy + yz + zx = t^2(\sin t\cos t + \sin t + \cos t)$$

ie.,
$$u = t^2(1/2\cdot\sin 2t + \sin t + \cos t)$$

Differentiating *w.r.t* t, we have,

$$\frac{du}{dt} = t^2(\cos 2t + \cos t - \sin t) + 2t(1/2\cdot\sin 2t + \sin t + \cos t)$$

$$\left(\frac{du}{dt}\right)_{t=\pi/4} = \frac{\pi^2}{16}\left(0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{\pi}{2}\left(\frac{1}{2}\cdot 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right) = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad \left(\frac{du}{dt}\right)_{t=\pi/4} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}} \qquad \ldots(2)$$

From (1) & (2) the result is verified.

---

41. *If* $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ *show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$ *and verify the result by direct differentiation.*

\>> $u = \sin^{-1}(x - y)$; $x = 3t$, $y = 4t^3$

$\{u \to (x, y) \to t\} \Rightarrow u \to t \quad \therefore \quad \dfrac{du}{dt}$ is the total derivative.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3}{\sqrt{\dfrac{1-9t^2+24t^4-16t^6}{(1-4t^2)^2}}}$$

Dividing $(1-9t^2+24t^4-16t^6)$ by $(1-8t^2+16t^4)$ we get $(1-t^2)$.

Thus $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$ . . . (1)

Now consider $u = \sin^{-1}(x-y) = \sin^{-1}(3t-4t^3)$

In order to differentiate u *w.r.t* 't' directly it is convenient by using the substitution $t = \sin\theta$ so that we have,

$$u = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(\sin 3\theta) = 3\theta$$

Thus $u = 3\theta$ and hence $\dfrac{du}{dt} = 3\dfrac{d\theta}{dt} = 3\dfrac{d}{dt}(\sin^{-1} t) = \dfrac{3}{\sqrt{1-t^2}}$

Thus $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$ . . . (2)

From (1) and (2) **the result is verified.**

Find $\dfrac{dy}{dx}$ in the following cases using partial derivatives.

42. $x^{2/3} + y^{2/3} = a^{2/3}$ 43. [illegible]

42. Let $u = f(x, y) = x^{2/3} + y^{2/3} - a^{2/3}$

$$\therefore \quad u_x = \frac{\partial u}{\partial x} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{2}{3}y^{2/3-1} = \frac{2}{3}y^{-1/3}$$

We have $\dfrac{dy}{dx} = -\dfrac{u_x}{u_y} = -\dfrac{2/3\,x^{-1/3}}{2/3\,y^{-1/3}} = \dfrac{-x^{-1/3}}{y^{-1/3}}$

Thus $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

$\therefore \quad u_x = \dfrac{\partial u}{\partial x} = a^x \log a - a^{(x+y)} \log a \cdot 1$

$u_y = \dfrac{\partial u}{\partial y} = a^y \log a - a^{x+y} \log a \cdot 1$

We have $\dfrac{dy}{dx} = -\dfrac{u_x}{u_y} = -\dfrac{\left[a^x \log a - a^{x+y} \log a \right]}{\left[a^y \log a - a^{x+y} \log a \right]}$

$= -\dfrac{\log a \,[\, a^x - a^{x+y} \,]}{\log a \,[\, a^y - a^{x+y} \,]} = -\dfrac{[\, a^x - a^{x+y} \,]}{[\, a^y - a^{x+y} \,]}$

Since $a^x + a^y = a^{x+y}$ by data, we get

$\dfrac{dy}{dx} = -\dfrac{(-a^y)}{-a^x}$ or $\dfrac{dy}{dx} = -a^{y-x}$

>> Let $u = f(x, y) = x^m y^n - (x+y)^{m+n}$

$\therefore \quad u_x = \dfrac{\partial u}{\partial x} = (m x^{m-1}) y^n - (m+n)(x+y)^{m+n-1}$

$u_y = \dfrac{\partial u}{\partial y} = x^m (n y^{n-1}) - (m+n)(x+y)^{m+n-1}$

Multiplying these partial derivatives by $(x+y)$ we get

$(x+y)\, u_x = m x^{m-1} y^n (x+y) - (m+n)(x+y)^{m+n}$

and $\quad (x+y)\, u_y = x^m (n y^{n-1})(x+y) - (m+n)(x+y)^{m+n}$

ie., $\quad (x+y)\, u_x = m x^{m-1} y^n (x+y) - (m+n) x^m y^n$

and $\quad (x+y)\, u_y = x^m (n y^{n-1})(x+y) - (m+n) x^m y^n$

ie., $\quad (x+y)\, u_x = x^{m-1} y^n \left[m(x+y) - (m+n) x \right]$

and $\quad (x+y)\, u_y = x^m y^{n-1} \left[n(x+y) - (m+n) y \right]$

ie., $\quad (x+y)\, u_x = x^{m-1} y^n (my - nx)$ $\quad \ldots (1)$

$$(x+y)\,u_y = x^m y^{n-1}(nx - my) \qquad \ldots (2)$$

But $\quad \dfrac{dy}{dx} = -\dfrac{u_x}{u_y} = -\dfrac{(x+y)\,u_x}{(x+y)\,u_y}$.

Using (1) and (2) we get,

$$\frac{dy}{dx} = \frac{-x^{m-1}y^n(my-nx)}{x^m y^{n-1}(nx-my)} = \frac{x^{m-1}y^n(nx-my)}{x^m y^{n-1}(nx-my)}.$$

Thus $\dfrac{dy}{dx} = \dfrac{y}{x}$ which is the desired result.

>> Here $\{ u \to (x, y)$ and $y \to x \} \Rightarrow u \to x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad \ldots (1)$$

Consider $u = \tan^{-1}(x/y)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+(x/y)^2}\cdot\frac{1}{y} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(x/y)^2}\cdot -\frac{x}{y^2} = \frac{y^2}{x^2+y^2}\cdot -\frac{x}{y^2} = -\frac{x}{x^2+y^2}$$

Also consider $x^2+y^2 = a^2$ and differentiating *w.r.t* x, we get

$$2x + 2y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y}$$

Using these results in (1) we have,

$$\frac{du}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\cdot\left(-\frac{x}{y}\right) = \frac{y^2+x^2}{(x^2+y^2)\,y} = \frac{1}{y}$$

Thus $\dfrac{du}{dx} = \dfrac{1}{y} = \dfrac{1}{\sqrt{a^2-x^2}}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad \ldots (1)$$

We have $u = \sqrt{x^2+y^2}$ $\quad\therefore\quad \frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$

Also $\quad \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$

Consider $x^3+y^3+3axy = 5a^2$ and differentiate *w.r.t* x

$\therefore \quad 3x^2+3y^2\frac{dy}{dx}+3a\left(x\frac{dy}{dx}+y\right) = 0$

or $\quad \frac{dy}{dx} = -\frac{(x^2+ay)}{(y^2+ax)}$

Now when $x = y = a$, $\frac{\partial u}{\partial x} = \frac{a}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}}$, $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{2}}$ and $\frac{dy}{dx} = -1$.

Substituting these values in (1) we get

$$\frac{du}{dx} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(-1) = 0$$

Thus when $x = y = a$, $\frac{du}{dx} = 0$

>> We have $u = e^{ax+by} f(ax-by)$

Let $\quad r = ax+by,\ s = ax-by$ so that $u = e^r f(s)$

Hence $\quad \{u \to (r, s) \to (x, y)\} \Rightarrow u \to x, y$

We have chain rules,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\ ;\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

ie., $\quad \frac{\partial u}{\partial x} = e^r f(s)\,a + e^r f'(s)\,a\ ;\ \frac{\partial u}{\partial y} = e^r f(s)\,b + e^r f'(s)(-b)$

ie., $\quad \frac{\partial u}{\partial x} = a\,e^r\left[f(s)+f'(s)\right]\ ;\ \frac{\partial u}{\partial y} = b\,e^r\left[f(s)-f'(s)\right]$

Now consider $b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y}$

$$= b \cdot ae^r \left[f(s)+f'(s)\right]+a \cdot be^r \left[f(s)-f'(s)\right]$$

$$= abe^r f(s)+abe^r f'(s)+abe^r f(s)-abe^r f'(s)$$

$$= 2ab\,e^r f(s) = 2abu$$

Thus $\quad b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2abu$

Remark : *Refering to Problem - 17, it may be observed that the same problem has been worked out directly.*

>> Here we need to convert the given function u into a composite function.

Let $\quad u = f(p, q, r)$ where $p = \dfrac{x}{y}$, $q = \dfrac{y}{z}$, $r = \dfrac{z}{x}$

ie., $\quad \{u \to (p, q, r) \to (x, y, z)\} \Rightarrow u \to x, y, z$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$ie., \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{1}{y} + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot\left(-\frac{z}{x^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \qquad \ldots(1)$$

Similarly by symmetry we can write,

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} \qquad \ldots(2)$$

$$z\frac{\partial u}{\partial z} = \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q} \qquad \ldots(3)$$

Adding (1), (2) and (3) we get $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

49. *If* $u = f(x-y, y-z, z-x)$ *show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

>> *(This example is similar to the previous example)*

Let $\quad u = f(p, q, r)$ where $p = x-y$, $q = y-z$, $r = z-x$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\,\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\,\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial x}$$

ie., $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot 1 + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}(-1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \qquad \ldots (1)$$

Similarly we have by symmetry,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \qquad \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \qquad \ldots (3)$$

Adding (1), (2) and (3) we get, $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

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50. If $z = f(x,$ [illegible]

(i) $(u+v)\dfrac{\partial z}{\partial x}$ [illegible] (ii) $(u+v$ [illegible] $= \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to u, v$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot v = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \qquad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot(-1) + \frac{\partial z}{\partial y}\cdot u = -\frac{\partial z}{\partial x} + u\,\frac{\partial z}{\partial y} \qquad \ldots (2)$$

(i) Consider R.H.S $= u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}$

$$= u\left[\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right] - v\left[\frac{-\partial z}{\partial x} + u\frac{\partial z}{\partial y}\right]$$

$= (u+v)\dfrac{\partial z}{\partial x}$ = L.H.S $\quad\therefore$ **R.H.S = L.H.S**

(ii) Consider R.H.S $= \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$

Adding (1) and (2) we have,

$\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = (u+v)\dfrac{\partial z}{\partial y}$ = L.H.S $\quad\therefore$ **R.H.S = L.H.S**

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\therefore \quad \frac{\partial z}{\partial u} = (2x)(e^u \sin v) + (2y)(e^u \cos v)$$

$$= 2e^u \left\{ e^u \sin v (\sin v) + e^u \cos v (\cos v) \right\}$$

$$= 2e^u \cdot e^u (\sin^2 v + \cos^2 v) = 2e^{2u}$$

$$\frac{\partial z}{\partial v} = (2x)(e^u \cos v) + (2y)(-e^u \sin v)$$

$$= 2e^u \left\{ e^u \sin v (\cos v) - e^u \cos v (\sin v) \right\} = 0$$

Thus as a composite function we have obtained

$$\frac{\partial z}{\partial u} = 2e^{2u} \; ; \; \frac{\partial z}{\partial v} = 0 \qquad \ldots (1)$$

Now consider $z = x^2 + y^2$ by direct substitution.

$z = (e^u \sin v)^2 + (e^u \cos v)^2 = e^{2u}(\sin^2 v + \cos^2 v) = e^{2u}$

Differentiating partially *w.r.t* u and also *w.r.t* v we get

$$\frac{\partial z}{\partial u} = 2e^{2u} \text{ and } \frac{\partial z}{\partial v} = 0 \qquad \ldots (2)$$

From (1) and (2) we conclude that **the result is verified.**

---

52. If $z = f(x, y)$ and $x = e^u \sin v$, $y = e^u \cos v$ prove that :

(i) $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

(ii) $\frac{\partial z}{\partial x} = e^{-u}\left\{\sin v \frac{\partial z}{\partial u} + \cos v \frac{\partial z}{\partial v}\right\}$

>> (i) $z \to \{(x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v) \qquad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(-e^u \sin v) \qquad \ldots (2)$$

Squaring and adding (1), (2) and collecting terms suitably we have,

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\frac{\partial z}{\partial x}\right)^2 [\sin^2 v + \cos^2 v] + e^{2u}\left(\frac{\partial z}{\partial y}\right)^2 [\cos^2 v + \sin^2 v]$$

$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u} \sin v \cos v - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u} \cos v \sin v$$

Thus $$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

(ii) Consider R.H.S $= e^{-u} \sin v \frac{\partial z}{\partial u} + e^{-u} \cos v \frac{\partial z}{\partial v}$

$$= e^{-u} \sin v \left\{\frac{\partial z}{\partial x} e^u \sin v + \frac{\partial z}{\partial y} e^u \cos v\right\} + e^{-u} \cos v \left\{\frac{\partial z}{\partial x} e^u \cos v - \frac{\partial z}{\partial y} e^u \sin v\right\}$$

$$= \frac{\partial z}{\partial x}\sin^2 v + \frac{\partial z}{\partial y}\sin v \cos v + \frac{\partial z}{\partial x}\cos^2 v - \frac{\partial z}{\partial y}\cos v \sin v$$

$$= \frac{\partial z}{\partial x}(\sin^2 v + \cos^2 v) = \frac{\partial z}{\partial x} = \text{L.H.S}$$

Thus R.H.S = L.H.S

Aliter : The problem can also be done by changing the independent variables x, y to u, v by elimination.

$$x = e^u \sin v, \quad y = e^u \cos v.$$

Squaring and adding : $x^2 + y^2 = e^{2u}$ or $\log(x^2 + y^2) = 2u$

Dividing, we have $\frac{x}{y} = \tan v$ or $v = \tan^{-1}(x/y)$

We can now write $z = f(u, v)$ where

$$u = \frac{1}{2}\log(x^2 + y^2), \quad v = \tan^{-1}(x/y)$$

ie., $\{z \to (u, v) \to (x, y)\} \Rightarrow z \to x, y$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

But $\quad \dfrac{\partial u}{\partial x} = \dfrac{1}{2}\cdot\dfrac{2x}{x^2+y^2} = \dfrac{e^u \sin v}{e^{2u}} = e^{-u}\sin v$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(x/y)^2}\cdot\frac{1}{y} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y}$$

$$= \frac{y}{x^2+y^2} = \frac{e^u \cos v}{e^{2u}} = e^{-u}\cos v$$

Thus $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}(e^{-u}\sin v) + \dfrac{\partial z}{\partial v}(e^{-u}\cos v)$

or $\quad \dfrac{\partial z}{\partial x} = e^{-u}\left(\sin v\dfrac{\partial z}{\partial u} + \cos v\dfrac{\partial z}{\partial v}\right)$

Thus **L.H.S = R.H.S**

>> $\{z \to (x, y) \to (r, \theta)\} \Rightarrow z \to (r, \theta)$

$$\therefore \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\ ;\ \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

ie., $\quad \dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta \qquad \ldots(1)$

and $\quad \dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}(r\cos\theta) = r\left[\dfrac{-\partial z}{\partial x}\sin\theta + \dfrac{\partial z}{\partial y}\cos\theta\right]$

or $\quad \dfrac{1}{r}\dfrac{\partial z}{\partial \theta} = \dfrac{-\partial z}{\partial x}\sin\theta + \dfrac{\partial z}{\partial y}\cos\theta \qquad \ldots(2)$

Squaring and adding (1), (2) and collecting suitable terms we have,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2[\cos^2\theta + \sin^2\theta] + \left(\frac{\partial z}{\partial y}\right)^2[\sin^2\theta + \cos^2\theta]$$
$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$\therefore \quad \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$ *ie.,* **R.H.S = L.H.S**

>> $\{(u, v) \to (x, y) \to (r, \theta)\} \Rightarrow (u,v) \to (r, \theta)$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \; ; \; \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \; ; \; \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

ie., $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad \ldots (1)$$

Next, by using the data we have,

$$\frac{\partial v}{\partial r} = \frac{-\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta \qquad \ldots (2)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

Next by using the data we have,

$$\frac{\partial v}{\partial \theta} = \frac{-\partial u}{\partial y}(-r\sin\theta) + \frac{\partial u}{\partial x}(r\cos\theta),$$

$$\therefore \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\sin\theta - \frac{\partial u}{\partial y}\cos\theta \qquad \ldots (3)$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \qquad \ldots (4)$$

By comparing that (1) & (4) and (2) & (3), the desired result follows,

Thus $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \qquad \ldots(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \qquad \ldots(2)$$

Consider R.H.S $= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$ and $(1)-(2)$ yields

$$\frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u}-e^{v}) = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

Thus $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ ie., R.H.S = L.H.S

>> $$u = x^{n} g\left(y/x,\ z/x\right) = x^{n} g(p,\ q) \qquad \ldots(1)$$

where $p = y/x$ and $q = z/x$

Now $$\frac{\partial u}{\partial x} = x^{n}\frac{\partial}{\partial x}\left[g(p,\ q)\right] + n\,x^{n-1} g(p,\ q)$$

$$\frac{\partial u}{\partial y} = x^{n}\frac{\partial}{\partial y}\left[g(p,\ q)\right]$$

$$\frac{\partial u}{\partial z} = x^{n}\frac{\partial}{\partial z}\left[g(p,\ q)\right]$$

We need to apply chain rule for computing the partial derivatives present in the R.H.S of the three equations.

$$\therefore \quad \frac{\partial}{\partial x}\left[g(p,\ q)\right] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial g}{\partial p}\left(-\frac{y}{x^{2}}\right) + \frac{\partial g}{\partial q}\left(-\frac{z}{x^{2}}\right)$$

$$\frac{\partial}{\partial y}\left[g(p,\ q)\right] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial y} = \frac{\partial g}{\partial p} \cdot \frac{1}{x} + \frac{\partial g}{\partial q} \cdot 0$$

$$\frac{\partial}{\partial z}\left[g(p,\ q)\right] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial z} = \frac{\partial g}{\partial p} \cdot 0 + \frac{\partial g}{\partial q} \cdot \frac{1}{x}$$

Hence, $$\frac{\partial u}{\partial x} = x^{n}\left[-\frac{y}{x^{2}}\frac{\partial g}{\partial p} - \frac{z}{x^{2}}\frac{\partial g}{\partial q}\right] + n\,x^{n-1} g(p,\ q)$$

or $$\frac{\partial u}{\partial x} = -x^{n-2}\left[y\frac{\partial g}{\partial p} + z\frac{\partial g}{\partial q}\right] + n\,x^{n-1} g(p,\ q) \qquad \ldots(2)$$

$$\frac{\partial u}{\partial y} = x^n \left[\frac{\partial g}{\partial p} \cdot \frac{1}{x} \right] = x^{n-1} \frac{\partial g}{\partial p} \qquad \ldots (3)$$

$$\frac{\partial u}{\partial z} = x^n \left[\frac{\partial g}{\partial q} \cdot \frac{1}{x} \right] = x^{n-1} \frac{\partial g}{\partial q} \qquad \ldots (4)$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ as a consequence of (2), (3) and (4)

$$= x \left\{ -x^{n-2} \left[y\frac{\partial g}{\partial p} + z\frac{\partial g}{\partial q} \right] + n\,x^{n-1} g(p,\ q) \right\} + y \cdot x^{n-1}\frac{\partial g}{\partial p} + z \cdot x^{n-1}\frac{\partial g}{\partial q}$$

$$= -x^{n-1} y\frac{\partial g}{\partial p} - x^{n-1} z\frac{\partial g}{\partial q} + nx^n g(p,\ q) + x^{n-1} y\frac{\partial g}{\partial p} + x^{n-1} z\frac{\partial g}{\partial q}$$

$$= n\left[x^n g(p,\ q) \right] = n\,u, \text{ by using (1).}$$

Thus $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\,u$, as required.

57. [illegible]

>> $\{u \to (v,\ w) \to (x,\ t)\} \Rightarrow \quad u \to (x,\ t)$

We have $v = x + ct,\ w = x - ct$, by data.

By chain rule, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$

ie., $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot 1 + \frac{\partial u}{\partial w} \cdot 1$ or $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = g\ (say)$

Next $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial g}{\partial x}$ and $g \to (v,\ w)$

Again by chain rule $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x}$

ie., $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot 1 + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot 1$$

ie., $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2} \qquad \ldots (1)$$

Now $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t}$

ie., $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial v} \cdot c + \dfrac{\partial u}{\partial w} \cdot (-c) = c\left[\dfrac{\partial u}{\partial v} - \dfrac{\partial u}{\partial w}\right] = h\,(say)$

Next $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial t}\right) = \dfrac{\partial h}{\partial t}$

Again by chain rule, $\dfrac{\partial h}{\partial t} = \dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial t} + \dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial t}$

ie., $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial v}\left\{c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\right\} \cdot c + \frac{\partial}{\partial w}\left\{c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\right\} \cdot (-c)$$

ie., $$\frac{\partial^2 u}{\partial t^2} = c^2\left\{\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2}\right\}$$

or $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2} \qquad \ldots (2)$$

Thus (1) – (2) will give us,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial v\,\partial w} \text{ as required.}$$

>> $\{w \to (u, v) \to (x, y)\} \Rightarrow \quad w \to x, y$

$u = e^x \sin y, \; v = e^x \cos y$

By chain rule, $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial x}$

ie., $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(e^x \sin y) + \frac{\partial w}{\partial v}(e^x \cos y)$$

or $$\frac{\partial w}{\partial x} = u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = g\,(say)$$

Next $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial x}\right) = \dfrac{\partial g}{\partial x}$ and $g \to u, v$

Again by chain rule, $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial x}$

ie., $$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial u}\left(u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v}\right)e^x \sin y + \frac{\partial}{\partial v}\left(u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v}\right)e^x \cos y$$

$$= \left(u\frac{\partial^2 w}{\partial u^2} + \frac{\partial w}{\partial u} + v\frac{\partial^2 w}{\partial u\,\partial v}\right)u + \left(u\frac{\partial^2 w}{\partial v\,\partial u} + v\frac{\partial^2 w}{\partial v^2} + \frac{\partial w}{\partial v}\right)v$$

$$\therefore \quad \frac{\partial^2 w}{\partial x^2} = u^2\frac{\partial^2 w}{\partial u^2} + u\frac{\partial w}{\partial u} + 2uv\frac{\partial^2 w}{\partial u\,\partial v} + v^2\frac{\partial^2 w}{\partial v^2} + v\frac{\partial w}{\partial v} \qquad \ldots (1)$$

Next $$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(e^x \cos y) + \frac{\partial w}{\partial v}(-e^x \sin y)$$

or $$\frac{\partial w}{\partial y} = v\frac{\partial w}{\partial u} - u\frac{\partial w}{\partial v} = h\ (say)$$

Again we have $$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial h}{\partial y}\ ;\ h \to u,\ v$$

$$\therefore \quad \frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial h}{\partial u}(e^x \cos y) + \frac{\partial h}{\partial v}(-e^x \sin y)$$

ie., $$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial u}\left(v\frac{\partial w}{\partial u} - u\frac{\partial w}{\partial v}\right)v + \frac{\partial}{\partial v}\left(v\frac{\partial w}{\partial u} - u\frac{\partial w}{\partial v}\right)(-u)$$

ie., $$\frac{\partial^2 w}{\partial y^2} = \left(v\frac{\partial^2 w}{\partial u^2} - u\frac{\partial^2 w}{\partial u\,\partial v} - \frac{\partial w}{\partial v}\right)v + \left(v\frac{\partial^2 w}{\partial v\,\partial u} + \frac{\partial w}{\partial u} - u\frac{\partial^2 w}{\partial v^2}\right)(-u)$$

ie., $$\frac{\partial^2 w}{\partial y^2} = v^2\frac{\partial^2 w}{\partial u^2} - 2uv\frac{\partial^2 w}{\partial u\,\partial v} - v\frac{\partial w}{\partial v} - u\frac{\partial w}{\partial u} + u^2\frac{\partial^2 w}{\partial v^2} \qquad \ldots (2)$$

$$\therefore \quad (1) + (2) \text{ gives, } \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u^2}(u^2 + v^2) + \frac{\partial^2 w}{\partial v^2}(u^2 + v^2)$$

Thus, $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$$

>> In polars we have $x = r\cos\theta$ and $y = r\sin\theta$

In order to find the two partial derivatives in the given equation as a composite function we need to have the composition in the form,

$$\{u \to (r,\ \theta) \to (x,\ y)\} \Rightarrow u \to x,\ y$$

$\therefore$ we have to express $r,\ \theta$ interms of x and y by simple familiar elimination.

$$x^2+y^2 = r^2 \ ; \ y/x = \tan\theta \ \text{ or } \ \theta = \tan^{-1}(y/x)$$

We do need the partial derivatives of r and θ *w.r.t* x and y while using the chain rule.

Hence, $r^2 = x^2+y^2$ gives $2r\dfrac{\partial r}{\partial x} = 2x$ and $2r\dfrac{\partial r}{\partial y} = 2y$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{or} \quad \frac{\partial r}{\partial x} = \cos\theta, \ \frac{\partial r}{\partial y} = \sin\theta$$

Also $\theta = \tan^{-1}(y/x)$ gives $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+(y/x)^2}\cdot -\dfrac{y}{x^2} = -\dfrac{y}{x^2+y^2}$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

or $\quad \dfrac{\partial\theta}{\partial x} = \dfrac{-\sin\theta}{r}, \quad \dfrac{\partial\theta}{\partial y} = \dfrac{\cos\theta}{r}$

Now by chain rule, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial\theta}\dfrac{\partial\theta}{\partial x}$

ie., $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cos\theta + \dfrac{\partial u}{\partial\theta}\left(\dfrac{-\sin\theta}{r}\right) = g\,(say) \ ; \ g \to r,\theta$

Also $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial g}{\partial x}$

Again by chain rule we have, $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial g}{\partial\theta}\dfrac{\partial\theta}{\partial x}$

ie.,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right]\cos\theta + \frac{\partial}{\partial\theta}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin\theta\right]\cos\theta$$

$$+ \left[\frac{\partial^2 u}{\partial\theta\,\partial r}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r} - \frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \qquad \ldots(1)$$

Next $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y}$, by chain rule.

ie., $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\sin\theta + \dfrac{\partial u}{\partial \theta}\dfrac{\cos\theta}{r} = h\,(say)\ ;\ h \to (r, \theta)$

$\therefore \quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial h}{\partial y}$

Again by chain rule we have $\dfrac{\partial h}{\partial y} = \dfrac{\partial h}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial h}{\partial \theta}\dfrac{\partial \theta}{\partial y}$

ie.,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right]\sin\theta + \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right]\left(\frac{\cos\theta}{r}\right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r\,\partial \theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r^2}\right]\sin\theta$$

$$+ \left[\frac{\partial^2 u}{\partial \theta\,\partial r}\sin\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right]\left(\frac{\cos\theta}{r}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial \theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \ldots (2)$$

Adding (1) and (2) we have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}(\cos^2\theta + \sin^2\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(\sin^2\theta + \cos^2\theta)$$

$$+ \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(\sin^2\theta + \cos^2\theta)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ transforms into

$\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ in the polar form.

1. If $u = e^{xy}$ show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{u}\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2\right\}$

2. If $u = e^{-x}(x\cos y + y\sin y)$ show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

3. If $u = \dfrac{1}{\sqrt{t}} e^{-x^2/4a^2 t}$ show that $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

4. If $e^{-z/x^2 - y^2} = x - y$, prove that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 - y^2$

5. If $u = f(x + iy) + g(x - iy)$ show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

6. If $z = f(y + x) + g(y + 2x) + 2e^{x+2y}$ show that $z_{xx} - 3z_{xy} + 2z_{yy} = 6e^{x+2y}$

7. If $z = xf_1(x + t) + f_2(x + t)$, show that $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\,\partial t} + \dfrac{\partial^2 z}{\partial t^2} = 0$

8. If $z = y^2 + 2f\left(\dfrac{1}{x} + \log y\right)$ show that $x^2\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2y^2$

9. If $u = \log\left(\dfrac{x^2 + y^2}{x + y}\right)$ verify that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$

10. If $u = \sinh^{-1}(x/y)$ verify that $u_{xy} = u_{yx}$

11. If $x = e^u \cos v$ and $y = e^u \sin v$ prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

    and verify that $\dfrac{\partial^2 v}{\partial x\,\partial y} = \dfrac{\partial^2 v}{\partial y\,\partial x}$

12. If $x = r\cos\theta$ and $y = r\sin\theta$, show that $\dfrac{\partial^2 r}{\partial x^2} \cdot \dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x\,\partial y}\right)^2$

13. If $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

    where $u = \dfrac{1}{r}$

14. If $u = r^m$ where $r^2 = x^2 + y^2 + z^2$,
show that $u_{xx} + u_{yy} + u_{zz} = m(m+1)r^{m-2}$

15. If $r^2 = x^2 + y^2 + z^2$, show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} = 0$

16. If $u = x^2 + y^2 + z^2$ where $x = e^{2t}$, $y = e^{2t}\cos t$, $z = e^{2t}\sin t$ find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.

17. If $u = \sin(x/y)$ where $x = e^t$, $y = t^2$ find $\frac{du}{dt}$ using the rule of composite functions.

18. If $z = \cos^{-1}(x - y)$ where $x = 4t^3$, $y = 3t$ prove that $\frac{dz}{dt} = -\frac{3}{\sqrt{1-t^2}}$ and also verify the result by direct differentiation.

19. If $x^3 + y^3 = 3xy$ find $\frac{dy}{dx}$ using partial derivatives.

20. Find $\frac{dy}{dx}$ given the relation $x^y + y^x = a$

21. If $u = x\log(xy)$ and $x^3 + y^3 - 3xy = 0$ find $\frac{du}{dx}$

22. If $z = f(x, y)$ where $x = e^u\cos v$, $y = e^u\sin v$ prove that
$(x+y)\frac{\partial z}{\partial u} + (x-y)\frac{\partial z}{\partial v} = e^{2u}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$

23. If $z = f(x, y)$ where $x = u^2 + v^2$, $y = 2uv$ show that
$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2(u+v)(u-v)\frac{\partial z}{\partial x}$

24. If $z = f(x, y)$ where $x = u^2 - v^2$, $y = 2uv$ show that
$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4(u^2 + v^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

25. If $u = f(x/z, y/z)$, prove that $x u_x + y u_y + z u_z = 0$

26. If $z = f(x, y)$ and $x = u - v$, $y = uv$, prove that
$(1+u)\frac{\partial z}{\partial u} + (1-v)\frac{\partial z}{\partial v} = 2(u+v)\frac{\partial z}{\partial y}$

27. If z is a function of x, y and u, v are connected by the relations $u = 3x + 4y$, $v = 3y - 4x$ show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 25\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

28. If $z = f(x, y)$ where $x = e^u \cos v,\ y = e^u \sin v$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2u}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

29. If $u = f(x, y)$ and $x = X\cos\alpha - Y\sin\alpha,\ y = X\sin\alpha + Y\cos\alpha$ prove that $u_{xx} + u_{yy} = u_{XX} + u_{YY}$

30. If $u = z^n f\left(\frac{x}{z}, \frac{y}{z}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

16. $8e^{4t}$ 17. $\frac{e^t(t-2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$ 19. $\frac{y - x^2}{y^2 - x}$

20. $\frac{-y(x^{y-1} + y^{x-1}\log y)}{x(y^{x-1} + x^{y-1}\log x)}$ 21. $\log(exy) + \frac{x(x^2 - y)}{y(x - y^2)}$

This topic is basically involved with the evaluation of determinants of second order and third order whose elements are represented by first order partial derivatives of two or three given functions / transformations.

The application of *Jacobians* is significant in the evaluation of double integrals of the form $\iint f(x, y)\,dx\,dy$ and triple integrals of the form $\iiint f(x, y, z)\,dx\,dy\,dz$ by transformation from one system (*set*) of coordinates (*variables*) to the other. The principle of evaluation is analogous with the evaluation of $\int f(x)\,dx$ by taking a suitable substitution.

[illegible]

Let u and v be functions of two independent variables x and y. The *Jacobian* (J) of u and v *w.r.t* x and y is symbolically represented and defined as follows

$$J\left(\frac{u,\ v}{x,\ y}\right) \text{ or } \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly if $u,\ v,\ w$ are functions of three independent variables $x,\ y,\ z$ then

$$J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) \text{ or } \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

\>> We have to find $J = \dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

But $\quad u = x + y + z,\ v = y + z,\ w = z$

$$\therefore \quad J = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding by the last row we get

$J = 1(1 - 0) = 1 \qquad$ Thus $J = 1$

\>> The definition of $J = \dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ is same as in the previous problem.

But $\quad u = x^2 + y^2 + z^2,\ v = xy + yz + zx,\ w = x + y + z$

Substituting for the partial derivatives we get

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}.$$

Expanding by the first row,

$$J = 2x\{(x+z)-(y+x)\} - 2y\{(y+z)-(y+x)\} + 2z\{(y+z)-(x+z)\}$$

$$= 2x(z-y) - 2y(z-x) + 2z(y-x)$$

$$= 2(xz - xy - yz + xy + yz - xz) = 0 \quad \text{Thus } J = 0$$

Aliter : *(By using properties of determinants)*

$$J = 2\begin{vmatrix} x & y & z \\ (y+z) & (x+z) & (y+x) \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ (x+y+z) & (x+y+z) & (x+y+z) \\ 1 & 1 & 1 \end{vmatrix} \cdots R_2 \to R_1 + R_2$$

$$J = 2(x+y+z)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \quad \because \text{ two rows are identical.}$$

>> $$J\left(\frac{u, v, w}{x_1, x_2, x_3}\right) = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial x_3} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{vmatrix}$$

where $u = \sqrt{x_1 x_2}$ $v = \sqrt{x_2 x_3}$, $w = \sqrt{x_3 x_1}$

$$= \begin{vmatrix} \dfrac{x_2}{2\sqrt{x_1 x_2}} & \dfrac{x_1}{2\sqrt{x_1 x_2}} & 0 \\ 0 & \dfrac{x_3}{2\sqrt{x_2 x_3}} & \dfrac{x_2}{2\sqrt{x_2 x_3}} \\ \dfrac{x_3}{2\sqrt{x_3 x_1}} & 0 & \dfrac{x_1}{2\sqrt{x_3 x_1}} \end{vmatrix}$$

$$= \frac{x_2}{2\sqrt{x_1 x_2}} \left\{ \frac{x_1 x_3}{4x_3\sqrt{x_1 x_2}} - 0 \right\} - \frac{x_1}{2\sqrt{x_1 x_2}} \left\{ 0 - \frac{x_2 x_3}{4x_3\sqrt{x_1 x_2}} \right\}$$

$$= \frac{x_1 x_2 x_3}{8x_1 x_2 x_3} + \frac{x_1 x_2 x_3}{8x_1 x_2 x_3} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \qquad \text{Thus } J = \frac{1}{4}$$

Aliter : $J = \dfrac{1}{2\sqrt{x_1 x_2}} \cdot \dfrac{1}{2\sqrt{x_2 x_3}} \cdot \dfrac{1}{2\sqrt{x_3 x_1}} \begin{vmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \end{vmatrix}$

$$= \frac{1}{8(x_1 x_2 x_3)} \left\{ x_2 (x_1 x_3 - 0) - x_1 (0 - x_2 x_3) \right\} = \frac{2x_1 x_2 x_3}{8x_1 x_2 x_3} = \frac{1}{4}$$

>> $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding by the last row, $J = 1(\rho\cos^2\phi + \rho\sin^2\phi) = \rho$

>> $$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$$

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi\ \ \ z = r\cos\theta$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

On expanding by the last row we get,

$$\cos\theta\left\{r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right\}$$

$$+ r\sin\theta\left\{r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right\}$$

$$= r^2\sin\theta\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin^3\theta\,(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta\cos^2\theta\cdot 1 + r^2\sin^3\theta\cdot 1 = r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$$

Thus $\dfrac{\partial(x,\,y,\,z)}{\partial(r,\,\theta,\,\phi)} = \boldsymbol{r^2\sin\theta}$

Aliter : Taking common factors r, $r\sin\theta$ respectively in the second and third columns we have

$$\frac{\partial(x,\,y,\,z)}{\partial(r,\,\theta,\,\phi)} = r\cdot r\sin\theta\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\ \sin\phi & \cos\theta\ \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by the last row we have

$$r^2\ \sin\theta\ \left\{\cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\ \sin^2\phi)\right.$$

$$\left. +\ \sin\theta\,(\sin\theta\cos^2\phi + \sin\theta\ \sin^2\phi)\right\}$$

$$= r^2\ \sin\theta\ \left\{\cos^2\theta\,(\cos^2\phi + \sin^2\phi)\ +\ \sin^2\theta\,(\cos^2\phi + \sin^2\phi)\right\}$$

$$= r^2\ \sin\theta\left\{\cos^2\theta\cdot 1 + \sin^2\theta\cdot 1\right\} = r^2\ \sin\theta\cdot 1 = \boldsymbol{r^2\sin\theta}$$

$$>>\quad \frac{\partial(x,\,y,\,z)}{\partial(u,\,v,\,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

It is evident that we should have $x,\ y,\ z$ in terms of $u,\ v,\ w$.

Consider $x + y + z = u\ \ \ldots(1)$, $\quad y + z = v\ \ \ldots(2)$, $\quad z = uvw\ \ \ldots(3)$

Using (2) in (1) we have, $x + v = u\quad \therefore\ \ x = u - v$

Also by using (3) in (2) we have, $y + uvw = v \quad \therefore\ y = v - uvw$

Thus the given data is modified into the form,

$$x = u - v,\ y = v - uvw,\ z = uvw$$

$$\therefore \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & (1-uw) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1\ \{(1-uw)\,uv - (uw)(-uv)\} + 1\ \{(-vw)(uv) - (vw)(-uv)\}$$

$$= uv - u^2 vw + u^2 vw - uv^2 w + uv^2 w = uv$$

Thus $J = uv$

Aliter : Adding R_3 to R_2 we get,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ vw & uw & uv \end{vmatrix}$$

On expanding by the second row we have

$$-0 + 1\,(uv - 0) - 0 = uv$$

>> By data $u = \frac{yz}{x},\ v = \frac{zx}{y},\ w = \frac{xy}{z}$

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right\}$$

$$-\frac{z}{x}\left\{\frac{z}{y}\left(\frac{-xy}{z^2}\right) - \frac{y}{z}\cdot\frac{x}{y}\right\} + \frac{y}{x}\left\{\frac{z}{y}\cdot\frac{x}{z} - \frac{y}{z}\left(\frac{-zx}{y^2}\right)\right\}$$

$$= \frac{-yz}{x^2}\left\{\frac{x^2}{yz} - \frac{x^2}{yz}\right\} - \frac{z}{x}\left\{\frac{-x}{z} - \frac{x}{z}\right\} + \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

$$= 0 + 1 + 1 + 1 + 1 = 4$$

Thus $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \mathbf{4}$

Aliter : We shall avoid denominators in every element by removal of $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ respectively from the first, second and the third row.

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{x^2 y^2 z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2 y^2 z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} \\ \cdots R_2 \to R_1 + R_2 \\ \cdots R_3 \to R_1 + R_3 \end{matrix}$$

$$= 1 \cdot \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -1(0-4) = \mathbf{4}$$

>> $\quad u = x + 3y^2 - z^3 \ , \ v = 4x^2 yz \ , \ w = 2z^2 - xy$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2 z & 4x^2 y \\ -y & -x & 4z \end{vmatrix}$$

It will be easier if the elements of the determinant are evaluated at $(1, -1, 0)$

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} \text{ at } (1, -1, 0) = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

Expanding by the second row we get, $-0 + 0 + 4(-1+6) = \mathbf{20}$

>> $\quad J\left(\dfrac{u, v}{x, y}\right)$ or $\dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Consider $u = x(1-r^2)^{-1/2}$

$$\therefore \quad \frac{\partial u}{\partial x} = x \cdot \frac{-1}{2}(1-r^2)^{-3/2} \cdot -2r\frac{\partial r}{\partial x} + 1 \cdot (1-r^2)^{-1/2}$$

$$= x(1-r^2)^{-3/2}\, r\, \frac{\partial r}{\partial x} + (1-r^2)^{-1/2}$$

But $\quad r^2 = x^2 + y^2.$

Differentiating partially *w.r.t* x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Similarly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now $\quad \dfrac{\partial u}{\partial x} = x(1-r^2)^{-3/2}\, r \cdot \dfrac{x}{r} + (1-r^2)^{-1/2}$

ie., $\quad \dfrac{\partial u}{\partial x} = (1-r^2)^{-3/2}\left\{x^2 + (1-r^2)\right\}$. But $x^2 - r^2 = -y^2$

$$\therefore \quad \frac{\partial u}{\partial x} = (1-r^2)^{-3/2}(1-y^2) \qquad \ldots (1)$$

Also $\quad \dfrac{\partial u}{\partial y} = x \cdot -\dfrac{1}{2}(1-r^2)^{-3/2} \cdot -2r\dfrac{\partial r}{\partial y}$

$$\therefore \quad \frac{\partial u}{\partial y} = x(1-r^2)^{-3/2}\, r \cdot \frac{y}{r} = xy(1-r^2)^{-3/2} \qquad \ldots (2)$$

Similarly we can obtain for $v = y(1-r^2)^{-1/2}$

$$\frac{\partial v}{\partial x} = xy(1-r^2)^{-3/2} \qquad \ldots (3)$$

$$\frac{\partial v}{\partial y} = (1-r^2)^{-3/2}(1-x^2) \qquad \ldots (4)$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} (1-r^2)^{-3/2}(1-y^2) & xy(1-r^2)^{-3/2} \\ xy(1-r^2)^{-3/2} & (1-r^2)^{-3/2}(1-x^2) \end{vmatrix}$$

Now taking $(1-r^2)^{-3/2}$ as a common factor in each row of the determinant we have,

$$\frac{\partial(u, v)}{\partial(x, y)} = (1-r^2)^{-3/2} \cdot (1-r^2)^{-3/2} \begin{vmatrix} (1-y^2) & xy \\ xy & (1-x^2) \end{vmatrix}$$

$$= (1-r^2)^{-3}\left\{(1-y^2)(1-x^2) - x^2y^2\right\}$$

$$= (1-r^2)^{-3}\left\{1 - x^2 - y^2 + x^2y^2 - x^2y^2\right\}$$

$$= (1-r^2)^{-3}\left\{1-(x^2+y^2)\right\} = (1-r^2)^{-3}(1-r^2) = (1-r^2)^{-2}$$

Thus $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{(1-r^2)^2}$

>> We have $x = a\cosh u\cos v,\ y = a\sinh u\sin v$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{vmatrix}$$

ie.,
$$= a^2(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v)$$

$$= \frac{a^2}{2}(\sinh^2 u \cdot 2\cos^2 v + \cosh^2 u \quad 2\sin^2 v)$$

But $1+\cos 2v = 2\cos^2 v$ and $1-\cos 2v = 2\sin^2 v$

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{a^2}{2}\left\{\sinh^2 u(1+\cos 2v) + \cosh^2 u(1-\cos 2v)\right\}$$

$$= \frac{a^2}{2}\left\{(\cosh^2 u + \sinh^2 u) - \cos 2v(\cosh^2 u - \sinh^2 u)\right\}$$

But $\cosh^2 u - \sinh^2 u = 1$ and $\cosh^2 u + \sinh^2 u = \cosh 2u$.

Thus, $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{a^2}{2}(\cosh 2u - \cos 2v)$

>> We have to find $\dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$.

Using the given data we have to solve for u and v interms of x and y.

By data $\quad u+v = e^x\cos y$ ... (1)

$\quad u-v = e^x\sin y$ ... (2)

$(1)+(2)$ gives : $2u = e^x(\cos y + \sin y)$

$(1)-(2)$ gives : $2v = e^x(\cos y - \sin y)$

ie., $u = \frac{e^x}{2}(\cos y + \sin y)$; $v = \frac{e^x}{2}(\cos y - \sin y)$

$\therefore$ $\frac{\partial u}{\partial x} = \frac{e^x}{2}(\cos y + \sin y)$, $\frac{\partial v}{\partial x} = \frac{e^x}{2}(\cos y - \sin y)$

$\frac{\partial u}{\partial y} = \frac{e^x}{2}(-\sin y + \cos y)$, $\frac{\partial v}{\partial y} = \frac{e^x}{2}(-\sin y - \cos y)$

Now $$\frac{\partial(u, v)}{\partial(x,} = \begin{vmatrix} \frac{e^x}{2}(\cos y + \sin y) & \frac{e^x}{2}(-\sin y + \cos y) \\ \frac{e^x}{2}(\cos y - \sin y) & -\frac{e^x}{2}(\sin y + \cos y) \end{vmatrix}$$

$$= \frac{e^x}{2} \cdot \frac{e^x}{2} \left\{ -(\cos y + \sin y)^2 - (\cos y - \sin y)^2 \right\}$$

$$= \frac{-e^{2x}}{4} \left\{ (1 + \sin 2y) + (1 - \sin 2y) \right\} = -\frac{e^{2x}}{2}$$

Thus $$\frac{\partial(u, v)}{\partial(x, y)} = \frac{-e^{2x}}{2}$$

1. $u = xy^2,\ v = yz^2,\ w = zx^2$
2. $u = x\cos y\cos z,\ v = x\cos y\sin z,\ w = x\sin y$
3. $u = \frac{y^2}{2x},\ v = \frac{x^2+y^2}{2x}$
4. $x = r\cos\theta\cos\phi,\ y = r\cos\theta\sin\phi,\ z = r\sin\theta$
5. $u = \frac{x}{y-z},\ v = \frac{y}{z-x},\ w = \frac{z}{x-y}$
6. If $u = xyz,\ v = xy+yz+zx,\ w = x+y+z$

   show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = (x-y)(y-z)(z-x)$

7. If $u = \frac{x}{\sqrt{1-r^2}}$, $v = \frac{y}{\sqrt{1-r^2}}$, $w = \frac{z}{\sqrt{1-r^2}}$ where $r = \sqrt{x^2+y^2+z^2}$, prove that $J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) = \frac{1}{(1-r^2)^{5/2}}$

8. Show that for the functions $x = u^2 - v^2$, $y = v^2 - w^2$, $z = w^2 - u^2$ $J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right) = 0$

9. Given that $ux_1 = 2x_2 x_3$, $vx_2 = 2x_3 x_1$, $wx_3 = 2x_1 x_2$ show that $\frac{\partial(u,\ v,\ w)}{\partial(x_1,\ x_2,\ x_3)} = 96$

10. If $x+y+z = u$, $y+z = uv$, $z = uvw$ then show that $\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)} = u^2 v$.

1. $9x^2y^2z^2$ 2. $-x^2\cos y$ 3. $-y/2x$

4. $r^2\cos\theta$ 5. 0

We denote $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h,\ b+\theta k)$ being the remainder after n terms.

Let $a+h = x$, $b+k = y$ ie., $h = x-a$, $k = y-b$

If x is close enough to a and y to b, h, k will be very small and $R_n \to 0$ as $n \to \infty$.

As $n \to \infty$ the number of terms increase indefinitely and we have an infinite series expansion of $f(x, y)$ in powers of $(x-a)$, $(y-b)$ referred to as the **Taylor's series of $f(x, y)$ about (a, b)** which is as follows.

$$f(x, y) = f(a, b) + \frac{1}{1!}\left\{(x-a)f_x(a, b) + (y-b)f_y(a, b)\right\}$$

$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right\} + \cdots \qquad \ldots (1)$$

In particular if $(a, b) = (0, 0)$, the series is called as **Taylor's series about the origin or Maclaurin's series for $f(x, y)$** given by

$$f(x, y) = f(0, 0) + \frac{1}{1!}\left\{x f_x(0, 0) + y f_y(0, 0)\right\}$$

$$+ \frac{1}{2!}\left\{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right\} + \ldots \qquad \ldots (2)$$

The remainder after n terms in this case will be

$$R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y) \quad \text{and} \quad R_n \to 0 \quad \text{as} \quad n \to \infty$$

Further if the Taylor's series of $f(x, y)$ is approximated to some terms upto a particular degree the resulting expression of $f(x, y)$ is called as the Taylor's polynomial.

[illegible]

1. Taylor's expansion [illegible]

>> The expansion of $f(x, y)$ about $(1, \pi/4)$ is given by

$$f(x, y) = f(1, \pi/4) + \left\{(x-1)f_x(1, \pi/4) + (y-\pi/4)f_y(1, \pi/4)\right\}$$

$$+ \frac{1}{2!}\left\{(x-1)^2 f_{xx}(1, \pi/4) + 2(x-1)(y-\pi/4)f_{xy}(1, \pi/4) + (y-\pi/4)^2 f_{yy}(1, \pi/4)\right\} + \cdots$$

The given function and its partial derivatives evaluated at $(1, \pi/4)$ is as follows.

$$f(x, y) = e^x \sin y \to e/\sqrt{2}$$

$$f_x = e^x \sin y \to e/\sqrt{2}$$

$$f_y = e^x \cos y \to e/\sqrt{2}$$

$$f_{xx} = e^x \sin y \to e/\sqrt{2}$$

$$f_{xy} = e^x \cos y \to e/\sqrt{2}$$

$$f_{yy} = -e^x \sin y \to -e/\sqrt{2}$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$e^x \sin y = \frac{e}{\sqrt{2}}\left[1 + \{(x-1) + (y-\pi/4)\} + \frac{1}{2!}\{(x-1)^2 + 2(x-1)(y-\pi/4) - (y-\pi/4)^2\}\right] + \cdots$$

>> The point (a, b) so as to obtain the expansion in powers of $(x-1)$ and $y+3$ is obviously $(1, -3)$. The associated expansion of $f(x, y)$ about $(1, -3)$ is given by

$$f(x, y) = f(1, -3) + \{(x-1)f_x(1, -3) + (y+3)f_y(1, -3)\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1, -3) + 2(x-1)(y+3)f_{xy}(1, -3) + (y+3)^2 f_{yy}(1, -3)\} + \cdots$$

The given function and its partial derivatives evaluated at $(1, -3)$ is as follows.

$$f(x, y) = xy^2 + x^2 y \to 6$$

$$f_x = y^2 + 2xy \to 3$$

$$f_y = 2xy + x^2 \to -5$$

$$f_{xx} = 2y \to -6$$

$$f_{xy} = 2y + 2x \to -4$$

$$f_{yy} = 2x \to 2$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$x y^2 + x^2 y = 6 + \{3(x-1) - 5(y+3)\} + \frac{1}{2!}\{-6(x-1)^2 - 8(x-1)(y+3) + 2(y+3)^2\} + \cdots$$

Taylor's expansion of $f(x, y)$ about $(0, 0)$ is given by

$$f(x, y) = f(0, 0) + \left\{ x f_x(0, 0) + y f_y(0, 0) \right\}$$
$$+ \frac{1}{2!}\left\{ x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0) \right\} + \cdots$$

The given function and its partial derivatives evaluated at $(0, 0)$ is as follows.

$$f(x, y) = e^x \log(1+y) \rightarrow 0$$
$$f_x = e^x \log(1+y) \rightarrow 0$$
$$f_y = e^x \cdot \frac{1}{1+y} \rightarrow 1$$
$$f_{xx} = e^x \log(1+y) \rightarrow 0$$
$$f_{xy} = e^x \cdot \frac{1}{1+y} \rightarrow 1$$
$$f_{yy} = e^x \cdot \frac{-1}{(1+y)^2} \rightarrow -1$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$\mathbf{e^x \log(1+y) = y + \frac{1}{2!}(2xy - y^2) + \cdots}$$

First, we briefly recapitulate the concept of maxima and minima for a function of one variable.

A function $f(x)$ is said to have a maximum value at a point $x = a$ if there exists a neighbourhood of the point 'a' say $(a+h)$, h is small, such that $f(a) > f(a+h)$. Similarly if $f(a) < f(a+h)$ then $f(x)$ is said to have a minimum value at $x = a$. $f(a)$ is said to be an extreme value of $f(x)$ if it is either a maximum or a minimum.

A necessary condition for $f(a)$ to be an extreme value of $f(x)$ is that $f'(a) = 0$ Also $f(a)$ is a maximum value of $f(x)$ if $f'(a) = 0$ and $f''(a) < 0$ & $f(a)$ is a minimum value of $f(x)$ if $f'(a) = 0$ and $f''(a) > 0$.

We discuss these concepts concerning a function of two variables.

Definition : A function $f(x, y)$ is said to have a **maximum value** at a point (a, b) if there exists a neighbourhood of the point (a, b) say $(a+h, b+k)$, h and k are small such that $f(a, b) > f(a+h, b+k)$

Similarly if $f(a, b) < f(a+h, b+k)$ then $f(x, y)$ is said to have a **minimum value** at (a, b). Also $f(a, b)$ is said to be an **extreme value** of $f(x, y)$ if it is a maximum value or a minimum value.

Necessary conditions : If $f(a, b)$ is an extreme value of $f(x, y)$ then it has to be an extreme value of the function of a single variable $f(x, b)$ and also of a function of a single variable $f(a, y)$. Hence the first derivatives of these two functions of a single variable must be zero as per the necessary condition of a function of a single variable.

ie., $\frac{d}{dx}[f(x, b)] = 0$ at $x = a$ and $\frac{d}{dy}[f(a, y)] = 0$ at $y = b$

In other words, these are the partial derivatives of $f(x, y)$ w.r.t. x and w.r.t. y at (a, b)

ie., $f_x(a, b) = 0$ **and** $f_y(a, b) = 0$.

These are the necessary conditions.

Suffecient conditions : Let $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Let us consider Taylor's theorem for $f(x, y)$ in the form,

$$f(a+h, b+k) = f(a, b) + \left\{ hf_x(a,b) + kf_y(a, b) \right\}$$
$$+ \frac{1}{2!}\left\{ h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b) \right\}$$

by neglecting other higher order terms for suffeciently small h and k.

Using $f_x(a, b) = 0$, $f_y(a, b) = 0$ and by denoting

$A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$ we have,

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(Ah^2 + 2Bhk + Ck^2)$$
$$= \frac{1}{2A}(A^2h^2 + 2ABhk + ACk^2)$$
$$= \frac{1}{2A}[(Ah + Bk)^2 + (AC - B^2)k^2]$$

ie., $f(a+h, b+k) - f(a, b) = \frac{\Delta}{2A}$ (*say*) ... (1)

where $\Delta = (Ah + Bk)^2 + (AC - B^2)k^2$

Clearly $\Delta > 0$ if $AC - B^2 > 0$ and $\Delta < 0$ if $AC - B^2 < 0$.

Further $f(a+h, b+k) - f(a, b) > 0$ if $AC - B^2 > 0, A > 0$

and $f(a+h, b+k) - f(a, b) < 0$ if $AC - B^2 > 0, A < 0$

$\Rightarrow$ $f(a+h, b+k) > f(a, b)$ if $AC - B^2 > 0$ and $A > 0$ and
$f(a+h, b+k) < f(a, b)$ if $AC - B^2 > 0$ and $A < 0$

or $f(a, b) > f(a+h, b+k)$ if $AC - B^2 > 0,\ A < 0$ and
$f(a, b) < f(a+h, b+k)$ if $AC - B^2 > 0,\ A > 0$

Taking a note of the definition for maxima-minima we conclude that,
$f(a, b)$ is a maximum value if $AC - B^2 > 0$ and $A < 0$

Conclusion : *Thus the necessary and the suffecient conditions for $f(x, y)$ to have a maximum value at (a, b) is $f_x(a, b) = 0,\ f_y(a, b) = 0$ and $AC - B^2 > 0,$ $A < 0.$ Also for a minimum value, $f_x(a, b) = 0,\ f_y(a, b) = 0$ and $AC - B^2 > 0,\ A > 0$*

Note : *If $AC - B^2 < 0$, $f(a, b)$ is not an extreme value. $f(x, y)$ is neither a maximum nor a minimum at the point (a, b) and the point is called a saddle point.*

If $AC - B^2 = 0$ then we have, $f(a+h, b+k) - f(a, b) = (Ah + Bk)^2/2A$

R.H.S is positive or negative according as $A > 0$ or $A < 0$. However if $Ah + Bk = 0$, then $h/k = -B/A$ and the case is doubtful. Further discussion is needed based on various geometrical factors.

- We have to first find the stationary points (x, y) such that $f_x = 0$ and $f_y = 0$
- We then find the second order partial derivatives
  $A = f_{xx},\ B = f_{xy},\ C = f_{yy}.$ We evaluate these at all the stationary points and also compute the corresponding value of $AC - B^2$
- (a) A stationary point (x_0, y_0) is a maximum point if
  $AC - B^2 > 0$ & $A < 0$. $f(x_0, y_0)$ is a maximum value
  (b) A stationary point (x_1, y_1) is a minimum point if
  $AC - B^2 > 0$ & $A > 0$. $f(x_1, y_1)$ is a minimum value.

Note : *We can overlook the cases of $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$*

\>> $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3\ ,\ f_y = 3y^2 - 12$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 - 3 = 0$ **and** $3y^2 - 12 = 0$ **or** $x^2 - 1 = 0$ **and** $y^2 - 4 = 0$

ie., $x = \pm 1,\ y = \pm 2$ $\therefore$ $(1, 2), (1, -2), (-1, 2), (-1, -2)$ are the stationary points. Let $A = f_{xx}\ ,\ B = f_{xy}\ ,\ C = f_{yy}$

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. pt.	Saddle pt.	Saddle pt.	Max. pt.

Maximum value of $f(x, y)$ **is**

$$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$$

Minimum value of $f(x, y)$ **is** $f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

\>> $f_x = 3x^2 + 3y^2 - 6x\ ;\ f_y = 6xy - 6y$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 + 3y^2 - 6x = 0$ or $x^2 + y^2 - 2x = 0$ ... (1)

$6xy - 6y = 0$ or $y(x - 1) = 0$ ... (2)

$\Rightarrow$ $y = 0,\ x = 1$ from (2).

From (1) if $y = 0$ then $x = 0, 2$ and if $x = 1$, $y = \pm 1$

$\therefore$ the stationary points are $(0, 0), (2, 0), (1, 1), (1, -1)$

Let $A = f_{xx},\ B = f_{xy},\ C = f_{yy}$

	(0, 0)	(2, 0)	(1, 1)	(1, −1)
$A = 6x - 6$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	−6
$C = 6x - 6$	−6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. pt.	Min. pt.	Saddle pt.	Saddle pt.

Maximum value of $f(x, y)$ is $f(0, 0) = 4$ and minimum value of $f(x, y)$ is $f(2, 0) = 0$

>> Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$f_x = 3x^2 + 3y^2 - 30x + 72 \ , \ f_y = 6xy - 30y$$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 + 3y^2 - 30x + 72 = 0$ or $x^2 + y^2 - 10x + 24 = 0$ . . . (1)

$6xy - 30y = 0$ or $y(x - 5) = 0$ . . . (2)

(2) gives us $y = 0$ and $x = 5$

Putting $y = 0$ in (1) we get $x^2 - 10x + 24 = 0$

or $(x - 4)(x - 6) = 0$ *ie.,* $x = 4, 6$

$\therefore$ (4, 0), (6, 0) are stationary points.

Putting $x = 5$ in (1) we get $y^2 - 1 = 0$ or $y = \pm 1$

$\therefore$ (5, 1), (5, −1) are also stationary points.

Let us examine these points for maxima and minima

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	(4, 0)	(6, 0)	(5, 1)	(5, −1)
$A = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	−6
$C = 6x - 30$	−6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. pt.	Min. pt.	Saddle pt.	Saddle pt.

Maximum value of $f(x, y)$ is $f(4, 0) = 64 - 240 + 288 = 112$

Minimum value of $f(x, y)$ is $f(6, 0) = 216 - 540 + 432 = 108$

>> Let $f(x, y) = axy - x^2 y - xy^2$

$f_x = ay - 2xy - y^2, \; f_y = ax - x^2 - 2xy$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

i.e., $y(a - 2x - y) = 0$ and $x(a - x - 2y) = 0$

Points (x, y) are obtained from the following set of equations.

(i) $y = 0, \; x = 0 \Rightarrow (x, y) = (0, 0)$

(ii) $y = 0, \; a - x - 2y = 0 \Rightarrow (x, y) = (a, 0)$

(iii) $a - 2x - y = 0, \; x = 0 \Rightarrow (x, y) = (0, a)$

(iv) $a - 2x - y = 0, \; a - x - 2y = 0 \Rightarrow (x, y) = (a/3, a/3)$ on solving.

We shall examine these points for maxima and minima.

Let $A = f_{xx}, \; B = f_{xy}, \; C = f_{yy}$

	$(0, 0)$	$(a, 0)$	$(0, a)$	$(a/3, a/3)$
$A = -2y$	0	0	$-2a$	$-2a/3$
$B = a - 2x - 2y$	a	$-a$	$-a$	$-a/3$
$C = -2x$	0	$-2a$	0	$-2a/3$
$AC - B^2$	$-a^2 < 0$	$-a^2 < 0$	$-a^2 < 0$	$a^2/3 > 0$
Conclusion	Saddle pt.	Saddle pt.	saddle pt.	Dependents on a

We note that $AC - B^2 > 0$ and $A = -2a/3$ is positive or negative according as a is negative or positive.

Hence $(a/3, a/3)$ is a minimum point if $a < 0$ and the same is a maximum point if $a > 0$

Now $f(a/3, a/3) = a^3/27$ **will be the extreme value of** $f(x, y)$ being minimum when $a < 0$ and maximum when $a > 0$.

>> Let $f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$

$f_x = 3x^2 - 63 + 12y, f_y = 3y^2 - 63 + 12x$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 - 63 + 12y = 0$ or $x^2 - 21 + 4y = 0$

$3y^2 - 63 + 12x = 0$ or $y^2 - 21 + 4x = 0$

On subtracting we have,

$$(x^2 - y^2) + 4(y - x) = 0 \text{ or } (x - y)(x + y - 4) = 0$$

$\therefore$ $x = y$ or $x + y = 4$

Putting $x = y$ in $x^2 - 21 + 4y = 0$ we have,

$$y^2 + 4y - 21 = 0 \text{ or } (y + 7)(y - 3) = 0 \text{ or } y = -7, 3$$

Since $x = y$, $(-7, -7)$ and $(3, 3)$ are stationary points.

Also $x + y$ [illegible] gives $x = 4 - y$ and $x^2 - 21 + 4y = 0$ becomes,

$(4 - y)^2 - 21 + 4y = 0$ or $y^2 - 4y - 5 = 0$ or $(y - 5)(y + 1) = 0$ or $y = 5, -1$.

Since $x = 4 - y$, $(-1, 5)$ and $(5, -1)$ are also stationary points.

Let us examine these four points for maxima and minima.

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(-7, -7)$	$(3, 3)$	$(-1, 5)$	$(5, -1)$
$A = 6x$	$-42 < 0$	$18 > 0$	-6	30
$B = 12$	12	12	12	12
$C = 6y$	-42	18	30	-6
$AC - B^2$	$1620 > 0$	$180 > 0$	$-324 < 0$	$-324 < 0$
Conclusion	Max.pt.	Min.pt.	Saddle pt.	Saddle pt.

Maximum value of $f(x, y) = f(-7, -7) = \mathbf{784}$

Minimum value of $f(x, y) = f(3, 3) = \mathbf{-216}$

[illegible]

>> $f_x = 4x^3 - 4(x - y)$; $f_y = 4y^3 + 4(x - y)$

We have to solve $f_x = 0$, $f_y = 0$ simultaneously.

$$x^3 - (x - y) = 0 \quad \ldots (1)$$

$$y^3 + (x - y) = 0 \quad \ldots (2)$$

From (1) $x - y = x^3$ and we shall use this in (2)

$\therefore \quad x^3 + y^3 = 0$ or $(x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow \quad y = -x \; ; \; x^2 - xy + y^2 = 0$

Using $y = -x$ in (1) we get, $x^3 - 2x = 0$ or $x(x^2 - 2) = 0$

$\therefore \quad x = 0, \; x = \pm\sqrt{2}. \quad y = -x, \Rightarrow y = 0, -\sqrt{2}, \sqrt{2}.$

Hence a set of stationary points are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

Also $x^2 - xy + y^2 = 0$ *ie.,* $x(x-y) + y^2 = 0$ and from *(1)* $x^3 = (x - y)$

$\therefore$ we have $x^4 + y^2 = 0$ which clearly shows that we do not have any real values satisfying the identity.

We shall now examine the stationary points for maxima - minima.

Let $\quad A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	-4	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$
Conclusion	-	Min. pt.	Min. pt.

Thus **minimum value** $= f(\sqrt{2}, -\sqrt{2})$ **or** $f(-\sqrt{2}, \sqrt{2}) = -8$

\>> $\quad f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

$$f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3, \qquad f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$$

Consider $f_x = 0$ and $f_y = 0$

ie., $\quad x^2 y^2 (3 - 4x - 3y) = 0$ **and** $x^3 y (2 - 2x - 3y) = 0$

$\Rightarrow \quad x = 0, \; y = 0, \; 4x + 3y = 3$ and $x = 0, \; y = 0, \; 2x + 3y = 2$

Let us form the pair of equations

$$\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\} \left.\begin{matrix} x=0 \\ 2x+3y=2 \end{matrix}\right\} \left.\begin{matrix} y=0 \\ 2x+3y=2 \end{matrix}\right\} \left.\begin{matrix} 4x+3y=3 \\ x=0 \end{matrix}\right\} \left.\begin{matrix} 4x+3y=3 \\ y=0 \end{matrix}\right\} \left.\begin{matrix} 4x+3y=3 \\ 2x+3y=2 \end{matrix}\right\}$$

The stationary points are $(0, 0)$ $(0, 2/3)$, $(1, 0)$ $(0, 1)$ $(3/4, 0)$ and $(1/2, 1/3)$

Also $A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y)$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6 - 8x - 9y)$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y \quad = 2x^3(1 - x - 3y)$

It is evident that either $A = 0$ or $C = 0$ or both A and C are zero in respect of all the stationary points except $(1/2, 1/3)$

When $A = 0$ or $C = 0$, $AC - B^2 < 0$ and we shall examine the nature of the point $(1/2, 1/3)$. At this point we get

$A = -1/9$, $B = -1/12$ $C = -1/8$

$\therefore \quad AC - B^2 = 1/72 - 1/144 = 1/144 > 0$. But $A = -1/9 < 0$

Hence $(1/2, 1/3)$ is a maximum point.

Thus **maximum value of $f(x, y) = f(1/2, 1/3) = 1/432$**

>> $z(x, y) = x^3 + y^3 - 3xy + 1$

$z_x = 3x^2 - 3y$; $z_y = 3y^2 - 3x$

Let $A = z_{xx}$, $B = z_{xy}$, $C = z_{yy}$ $\therefore$ $A = 6x$, $B = -3$, $C = 6y$

Now, at $(1, 1)$ $z_x = 0$ and $z_y = 0$

Also $A = 6$, $B = -3$, $C = 6$. $\therefore$ $AC - B^2 = 27 > 0$

Now at $(1, 1)$, $z_x = 0$ and $z_y = 0$. $AC - B^2 > 0$, $A = 6 > 0$

$\therefore$ $z(x, y)$ at $(1, 1)$ satisfy the necessary and suffecient conditions for minima.

Thus **$z(x, y)$ is minimum at $(1, 1)$**

>> $f(x, y) = 1 + \sin(x^2 + y^2)$

$f_x = 2x\cos(x^2 + y^2)$, $f_y = 2y\cos(x^2 + y^2)$

We shall find points such that $f_x = 0$, $f_y = 0$

ie., $2x\cos(x^2 + y^2) = 0$ and $2y\cos(x^2 + y^2) = 0$

$\therefore \quad x = 0,\ y = 0$ and $(0, 0)$ is the stationary point.

$$A = f_{xx} = -4x^2 \sin(x^2 + y^2) + 2\cos(x^2 + y^2)$$

$$B = f_{xy} = -4xy \sin(x^2 + y^2)$$

$$C = f_{yy} = -4y^2 \sin(x^2 + y^2) + 2\cos(x^2 + y^2)$$

At $(0, 0)$: $A = 2$, $B = 0$, $C = 2 \quad \therefore \quad AC - B^2 = 4 > 0$

Since $AC - B^2 > 0$, $A = 2 > 0$, $(0, 0)$ is a minimum point and the **minimum value** of $f(x, y) = f(0, 0) = 1$

>> $\quad f(x, y) = \sin x + \sin y + \sin(x + y)$

$$f_x = \cos x + \cos(x + y) \ ; \ f_y = \cos y + \cos(x + y)$$

Let us consider $f_x = 0$ and $f_y = 0$

ie., $\quad \cos x + \cos(x + y) = 0$ and $\cos y + \cos(x + y) = 0$

or $\quad \cos(x + y) = -\cos x$ and $\cos(x + y) = -\cos y$

Thus $\quad -\cos x = -\cos y \quad \Rightarrow \quad x = y$

Putting $y = x$ in $\cos x + \cos(x + y) = 0$ we get $\cos x + \cos 2x = 0$

ie., $\quad \cos x + 2\cos^2 x - 1 = 0$

ie., $\quad 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

$\Rightarrow \quad \cos x = 1/2$, $\cos x = -1 \quad \therefore \ x = \pi/3$ and $x = \pi$

Thus $(\pi/3, \pi/3)$ and (π, π) are the stationary points.

Let $\quad A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

ie., $\quad A = -\sin x - \sin(x + y),\ B = -\sin(x + y),\ C = -\sin y - \sin(x + y)$

At $\quad (\pi/3, \pi/3)$: $A = -\sin(\pi/3) - \sin(2\pi/3) = -(\sqrt{3}/2) - (\sqrt{3}/2)$

$$A = -\sqrt{3},\ B = -(\sqrt{3}/2),\ C = -\sqrt{3}$$

$\therefore \quad AC - B^2 = 3 - (3/4) = 9/4 > 0 \ ; \ A = -\sqrt{3} < 0$

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and the maximum value is $\sin(\pi/3) + \sin(\pi/3) + \sin(2\pi/3) = \sqrt{3}/2 + \sqrt{3}/2 + \sqrt{3}/2$

Thus **maximum value** $= 3\sqrt{3}/2$

At (π, π) : $A = 0,\ B = 0,\ C = 0$ and hence the case needs further investigation.

1. $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
2. $x^3 y^2\ (12 - 3x - 4y)$
3. $x^3 y^2\ (12 - x - y)$
4. $x^2 y\ (x + 2y - 4)$
5. $x^3 + y^3 - 3axy,\quad a > 0$

1. Min. value $= 0$ at $(2,\ 0)$ and max $\cdot$ value $= 4$ at $(0,\ 0)$
2. max $\cdot$ value $= 8$ at $(2,\ 1)$
3. Max. value $= 6922$ at $(6,\ 4)$
4. Min. value $= -2$ at $(2,\ 1/2)$
5. Min. value $= -a^3$ at $(a,\ a)$

Many practical situations like computation involving various numerical quantities, taking measurements or readings, changing the scale etc. are susceptible to errors of various kinds based on various factors. Obviously it is going to have an impact on the final result which sometimes may be negligible or significant too. Further many numerical calculations will force us to approximate and in such cases any computed result will contain errors.

This topic gives an insight to such situations only, though the topic can be discussed still widely concerning the types of errors, error minimization etc.

We know that if y is dependent on x, that is if $y = f(x)$ then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This is equivalent to saying that the quantities $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ are equal if δx is small enough.

$$\therefore \quad \delta y = \frac{dy}{dx}\,\delta x$$

or $\quad \delta y = f'(x)\,\delta x \qquad \ldots (1)$

The value δy is called the error in y due to an error δx in x.

Further in the computation of a numerical quantity, if x_0 is taken as an approximate value of x, that is when x is approximated to x_0

then $\delta x = |x - x_0|$ is called as the *absolute error in* x.

$(\delta x/x)$ is called as the *relative error in* x.

$(\delta x/x) \times 100$ is called as the *percentage error in* x.

Also if $z = f(x, y)$ then we have,

$$\delta z = \frac{\partial z}{\partial x}\,\delta x + \frac{\partial z}{\partial y}\,\delta y$$

The associated relative and percentage errors are respectively $(\delta z/z)$ and $(\delta z/z) \times 100$.

[illegible]

- We prefer to take logarithms first for the given expression and then take the differential as it would always give terms of the form $(\delta x/x)$.
- We can as well multiply with 100 if the given data involves percentage errors.
- The following formulae will be useful.

Area Volume Surface area

1. Circle $[x^2 + y^2 = r^2] \ldots$ Area $= \pi r^2$
2. Ellipse $[x^2/a^2 + y^2/b^2 = 1] \ldots$ Area $= \pi ab$

Name of the Solid	Volume	Lateral or Curved Surface Area	Total Surface Area
1. Cube	a^3	$4a^2$	$6a^2$
2. Cuboid	lbh	$2(l+b)h$	$2(lb+bh+lh)$
3. Cylinder	$\pi r^2 h$	$2\pi rh$	$2\pi r(r+h)$
4. Cone	$1/3 \cdot \pi r^2 h$	πrl	$\pi r(r+l)$
5. Sphere	$4/3 \cdot \pi r^3$	-	$4\pi r^2$

where in the case of cone l is the slant height connected by the relation $l^2 = r^2 + h^2$

Note : The following results similar to that of differentials will also be useful.

(i) $\delta(cu) = c\,\delta u$ (ii) $\delta(u \pm v) = \delta u \pm \delta v$

(iii) $\delta(uv) = u\,\delta v + v\,\delta u$ (iv) $\delta\left(\dfrac{u}{v}\right) = \dfrac{v\,\delta u - u\,\delta v}{v^2}$

>> $PV^2 = K$ by data. Also $\dfrac{\delta P}{P} = 0.05$ and $\dfrac{\delta V}{V} = 0.025$

$\Rightarrow \quad \log P + 2\log V = \log K$

$\Rightarrow \quad \delta(\log P) + 2\delta(\log V) = \delta(\log K)$

ie., $\quad \dfrac{1}{P}\,\delta P + 2 \cdot \dfrac{1}{V}\,\delta V = \dfrac{1}{K}\,\delta K$

ie., $\quad 0.05 + 2(0.025) = \dfrac{\delta K}{K}$ or $\dfrac{\delta K}{K} = 0.1$

$\therefore \quad \dfrac{\delta K}{K} \times 100 = (0.1) \times 100 = 10$

Thus the error in K is 10%.

>> (i) $T = 2\pi\sqrt{l/g}$, g = constant, $\dfrac{\delta l}{l} \times 100 = 3$

$\Rightarrow \quad \log T = \log 2\pi + \dfrac{1}{2}(\log l - \log g)$

$\Rightarrow \quad \delta(\log T) = \delta(\log 2\pi) + \dfrac{1}{2}\delta(\log l) - \dfrac{1}{2}\delta(\log g)$

ie., $\quad \dfrac{\delta T}{T} = 0 + \dfrac{1}{2}\,\dfrac{\delta l}{l} - 0$

or $\quad \dfrac{\delta T}{T} \times 100 = \dfrac{1}{2}\left(\dfrac{\delta l}{l} \times 100\right) = \dfrac{1}{2}(3) = 1.5$

Thus **the error in T = 1.5%.**

(ii) If g is not a constant we have,

$$\frac{\delta T}{T} \times 100 = \frac{1}{2}\left(\frac{\delta l}{l} \times 100\right) - \frac{1}{2}\left(\frac{\delta g}{g} \times 100\right)$$

The error in T will be **maximum** if the error in l is positive and the error in g is negative (or *vice-versa*) as the difference in errors converts into a sum.

$$\therefore \quad \max\left(\frac{\delta T}{T} \times 100\right) = \frac{1}{2}(+1) - \frac{1}{2}(-3) = 2$$

Thus **the maximum error in T is 2%.**

\>> Consider $c = k \tan\theta$. k is taken as a constant.

$$\Rightarrow \quad \log c = \log k + \log(\tan\theta)$$

$$\Rightarrow \quad \delta(\log c) = \delta(\log k) + \delta \log(\tan\theta)$$

ie., $$\frac{1}{c}\,\delta c = 0 + \frac{\sec^2\theta}{\tan\theta}\,\delta\theta$$

ie., $$\frac{\delta c}{c} = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos^2\theta}\,\delta\theta \quad \text{or} \quad \frac{\delta c}{c} = \frac{\delta\theta}{\sin\theta\cos\theta}$$

ie., $$\frac{\delta c}{c} = \frac{2}{\sin 2\theta}\,\delta\theta$$

The relative error in c being $\delta c/c$ is minimum when the denominator of the R.H.S is maximum and the maximum value of a sine function is 1.

$$\therefore \quad \sin 2\theta = 1 \Rightarrow 2\theta = 90^0 \text{ or } \theta = 45^0.$$

Thus the relative error in c is minimum when $\theta = 45^{\circ}$

84. *If $T = \frac{1}{2}mv^2$ is the kinetic energy, find approximately the change in T as m changes from* 49 *to* 49.5 *and v changes from* 1600 *to* 1590.

\>> We have by data $T = \frac{1}{2}mv^2$ and

$$m = 49,\ m + \delta m = 49.5 \ \therefore\ \delta m = 0.5$$

$$v = 1600,\ v + \delta v = 1590 \ \therefore\ \delta v = -10$$

We have to find δT. (*logarithm is not required*)

$\therefore \quad \delta T = \frac{1}{2}\delta(mv^2)$

$= \frac{1}{2}\left\{ m(2v\,\delta v) + \delta m \cdot v^2 \right\}$

ie., $= \frac{1}{2}\left\{ (49)(2)(1600)(-10) + (0.5)(1600)^2 \right\} = -1,44,000$

Thus the change in $T = \delta T = -1,44,000$.

>> $p\,v^{1.4}$ = constant = c (*say*), by data.

$\Rightarrow \quad \log p + 1.4 \log v = \log c$

$\Rightarrow \quad \delta(\log p) + 1.4\,\delta(\log v) = \delta(\log c)$

ie., $\frac{\delta p}{p} + 1.4\left(\frac{\delta v}{v}\right) = 0$; But $\frac{\delta v}{v} \times 100 = -\frac{1}{2}$, by data.

$\therefore \quad \frac{\delta p}{p} \times 100 + 1.4\left(\frac{\delta v}{v} \times 100\right) = 0$ or $\frac{\delta p}{p} \times 100 = +0.7$

Thus the percentage increase in pressure = 0.7

>> Let x be the deflection and we have by data

$x \propto \frac{wl^3}{d^4} \Rightarrow x = k\,\frac{wl^3}{d^4}$, k being a constant.

$\therefore \quad \log x = \log k + \log w + 3 \log l - 4 \log d$

$\Rightarrow \quad \delta(\log x) = \delta(\log k) + \delta(\log w) + 3\delta(\log l) - 4\delta(\log d)$

$\Rightarrow \quad \frac{\delta x}{x} = 0 + \frac{\delta w}{w} + 3\left(\frac{\delta l}{l}\right) - 4\left(\frac{\delta d}{d}\right)$

$\Rightarrow \quad \frac{\delta x}{x} \times 100 = \frac{\delta w}{w} \times 100 + 3\left(\frac{\delta l}{l} \times 100\right) - 4\left(\frac{\delta d}{d} \times 100\right)$

But $\frac{\delta w}{w} \times 100 = 5,\ \frac{\delta l}{l} \times 100 = 4,\ \frac{\delta d}{d} \times 100 = 3$, by data.

$\therefore \quad \frac{\delta x}{x} \times 100 = 5+3(4)-4(3) = 5$

Thus the percentage increase in the deflection = 5

>> By data, $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$ and $\delta u = \delta v = e$

$\Rightarrow \quad 2\delta\left(\frac{1}{f}\right) = \delta\left(\frac{1}{v}\right) - \delta\left(\frac{1}{u}\right)$

ie., $\quad 2\left(-\frac{1}{f^2}\,\delta f\right) = -\frac{1}{v^2}\,\delta v + \frac{1}{u^2}\,\delta u$

ie., $\quad \frac{1}{f}\,\delta f = \frac{f}{2}\left[\frac{1}{v^2}\,\delta v - \frac{1}{u^2}\,\delta u\right]$

ie., $\quad \frac{\delta f}{f} = e \cdot \frac{f}{2}\left(\frac{1}{v}+\frac{1}{u}\right)\left(\frac{1}{v}-\frac{1}{u}\right)$

ie., $\quad \frac{\delta f}{f} = e \cdot \frac{f}{2}\left(\frac{1}{v}+\frac{1}{u}\right)\frac{2}{f}$

Thus the relative error in $f = e\left(\frac{1}{u}+\frac{1}{v}\right)$

>> $\frac{1}{f} = \frac{1}{p}+\frac{1}{q}$; $p = q = 20$; $\delta p = \delta q = 0.5$

$\Rightarrow \quad \delta\left(\frac{1}{f}\right) = \delta\left(\frac{1}{p}\right)+\delta\left(\frac{1}{q}\right)$

ie., $\quad -\frac{1}{f^2}\ \delta f = -\frac{1}{p^2}\,\delta p - \frac{1}{q^2}\,\delta q$

or $\delta f = f^2 \left[\frac{1}{p^2} \delta p + \frac{1}{q^2} \delta q \right] = \frac{2f^2}{p^2} \delta p \quad \because \; p = q, \; \delta p = \delta q$

Also $\frac{1}{f} = \frac{1}{20} + \frac{1}{20} = 0.1$ or $f = 10$

$\therefore \quad \delta f = \frac{2(10)^2}{(20)^2}(0.5) = 0.25$

Thus the maximum error in $f = 0.25$

>> If l and b denotes the length and the breadth of a rectangle then the area (A) is given by

$A = lb$; Also by data $\frac{\delta l}{l} \times 100 = 1 = \frac{\delta b}{b} \times, 100$

We have $\log A = \log l + \log b$

$\Rightarrow \quad \delta(\log A) = \delta(\log l) + \delta(\log b)$

ie., $\frac{\delta A}{A} = \frac{\delta l}{l} + \frac{\delta b}{b}$ or $\frac{\delta A}{A} \times 100 = \frac{\delta l}{l} \times 100 + \frac{\delta b}{b} \times 100$

$\therefore \quad \frac{\delta A}{A} \times 100 = 1 + 1 = 2$

Thus the error in the area is 2%

>> For the ellipse $x^2/a^2 + y^2/b^2 = 1$ the area (A) is given by πab where $2a$ and $2b$ are the lengths of the major and minor axis.

Let $2a = x$ and $2b = y$.

By data $\frac{\delta x}{x} \times 100 = 1, \; \frac{\delta y}{y} \times 100 = 1.$

$$A = \pi ab = \pi \cdot \frac{x}{2} \cdot \frac{y}{2} = \frac{\pi}{4} xy$$

$\therefore \quad \log A = \log(\pi/4) + \log x + \log y$

$\Rightarrow \quad \delta(\log A) = \delta \log(\pi/4) + \delta(\log x) + \delta(\log y)$

ie., $\frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y}$ or $\frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$

$\therefore$ $\frac{\delta A}{A} \times 100 = 1 + 1 = 2$

Thus error in the area = 2%

>> We know the volume V of a right circular cone is given by

$V = \frac{1}{3}\pi r^2 h$. By data $\frac{\delta h}{h} \times 100 = 2$, $\frac{\delta r}{r} \times 100 = 1$

$\Rightarrow$ $\log V = \log(\pi/3) + 2\log r + \log h$

$\Rightarrow$ $\delta(\log V) = \delta \log(\pi/3) + 2\delta(\log r) + \delta(\log h)$

ie., $\frac{\delta V}{V} = 2\frac{\delta r}{r} + \frac{\delta h}{h}$ or $\frac{\delta V}{V} \times 100 = 2\frac{\delta r}{r} \times 100 + \frac{\delta h}{h} \times 100$

$\therefore$ $\frac{\delta V}{V} \times 100 = 2(1) + 2 = 4$

Thus the error in the volume = 4%

>> The volume of the sphere is given by $V = \frac{4}{3}\pi r^3$

$\Rightarrow$ $\log V = \log(4\pi/3) + 3\log r$; $\frac{\delta r}{r} \times 100 = 1 \cdot 25$ by data.

$\Rightarrow$ $\delta(\log V) = 0 + 3\delta(\log r)$

ie., $\frac{\delta V}{V} = 3 \cdot \frac{\delta r}{r}$ or $\frac{\delta V}{V} \times 100 = 3 \cdot \frac{\delta r}{r} \times 100$

$\therefore$ $\frac{\delta V}{V} \times 100 = 3(1.25) = 3.75$

Thus the error in the volume = 3.75%

The surface area of the sphere $(S) = 4\pi r^2$

$\Rightarrow$ $\log S = \log(4\pi) + 2\log r$

$\Rightarrow$ $\delta(\log S) = 2\delta(\log r)$

ie., $\frac{\delta S}{S} = 2 \cdot \frac{\delta r}{r}$ or $\frac{\delta S}{S} \times 100 = 2 \frac{\delta r}{r} \times 100$

$\therefore \quad \frac{\delta S}{S} \times 100 = 2(1.25) = 2.5$

Thus the error in the surface area = 2.5%

>> The volume (V) of a right circular cylinder is given by $V = \pi r^2 h$. If D is the diameter then $r = D/2$

$\therefore \quad V = \pi \frac{D^2}{4} \cdot h$ *ie.*, $V = \frac{\pi}{4} D^2 h$

By data, $D = 4.5$, $\delta D = 0.1$, $h = 8.25$, $\delta h = 0.1$

Now $\delta V = \frac{\pi}{4} \delta(D^2 h)$

ie., $\delta V = \frac{\pi}{4}\left(D^2 \cdot \delta h + 2D\,\delta D \cdot h\right)$

$= \frac{\pi}{4}\{(4.5)^2(0.1) + 2(4.5)(0.1)(8.25)\} = 7.42$

Thus the error in the volume = 7.42cm^3

Also the lateral surface area (S) of the cylinder is given by $S = 2\pi rh$ where $r = D/2$.

ie., $S = \pi Dh$

$\therefore \quad \delta S = \pi(D\,\delta h + h\,\delta D)$

$= \pi\left(4.5 \times 0.1 + 8.25 \times 0.1\right)$

ie., $\delta S = (0.1)\pi(12.75) = 4.0055 = 4$

Thus the error in the surface area = 4cm^2

>> Volume of the sphere $(V) = \frac{4}{3}\pi r^3$

$$\therefore \quad \delta V = \frac{4\pi}{3}\delta(r^3) = \frac{4\pi}{3}(3r^2\,\delta r) = 4\pi r^2\,\delta r$$

The relative error in the volume :

$$\frac{\delta V}{V} = \frac{4\pi r^2\,\delta r}{4/3\cdot\pi r^3} = 3\,\frac{\delta r}{r}. \quad \text{But } r = 10,\ \delta r = 0.02.$$

$$\therefore \quad \frac{\delta V}{V} = \frac{3(0.02)}{10} = 0.006$$

Thus the relative error in the volume = 0.006 cm^3

>> By data $r = 6,\ h = 18,\ \delta r = \delta h = 0.15$

We have $V = \pi r^2 h$, being the volume of the right circular cylinder.

$$\Rightarrow \quad \log V = \log\pi + 2\log r + \log h$$

$$\Rightarrow \quad \delta(\log V) = \delta(\log\pi) + 2\delta(\log r) + \delta(\log h)$$

$$ie., \quad \frac{\delta V}{V} = 0 + 2\,\frac{\delta r}{r} + \frac{\delta h}{h} = \delta r\left(\frac{2}{r} + \frac{1}{h}\right) \quad \because\ \delta r = \delta h$$

$$\therefore \quad \frac{\delta V}{V} = 0.15\left[\frac{2}{6} + \frac{1}{18}\right] = 0.15\left(\frac{7}{18}\right) = 0.0583$$

$$\Rightarrow \quad \frac{\delta V}{V} \times 100 = 5.83$$

Thus the percentage error in the volume = 5.83

We have $S = 2\pi r h$, being the surface area of the cylinder.

$$\Rightarrow \quad \log S = \log(2\pi) + \log r + \log h$$

$$\Rightarrow \quad \delta(\log S) = 0 + \delta(\log r) + \delta(\log h)$$

ie., $\frac{\delta S}{S} = \frac{\delta r}{r} + \frac{\delta h}{h} = \delta r\left(\frac{1}{r} + \frac{1}{h}\right) = 0.15\left(\frac{1}{6} + \frac{1}{18}\right) = 0.0333$

$\therefore \quad \frac{\delta S}{S} \times 100 = 3.33$

Thus the percentage error in the surface area = 3.33

>> Let $l,\ b,\ h$ respectively be the length, breadth and height of the pile of bricks so that the volume (V) of the pile is

$$V = lbh$$

$\Rightarrow \quad \log V = \log l + \log b + \log h$

$\Rightarrow \quad \delta(\log V) = \delta(\log l) + \delta(\log b) + \delta(\log h)$

ie., $\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta h}{h}$ . . . (1)

Since the tape is stretched by 1%, the error in $l,\ b,\ h$ is 1%

ie., $\frac{\delta l}{l} \times 100 = \frac{\delta b}{b} \times 100 = \frac{\delta h}{h} \times 100 = 1$

or $\frac{\delta l}{l} = \frac{\delta b}{b} = \frac{\delta h}{h} = \frac{1}{100} = 0.01$

$\therefore$ (1) becomes, $\frac{\delta V}{V} = 0.01 + 0.01 + 0.01 = 0.03$ . . . (2)

But we have by data $l = 2,\ b = 15,\ h = 1.2$

$\therefore \quad V = lbh = 2 \times 15 \times 1.2 = 36$ cubic metres.

Hence (2) gives, $\delta V = 0.03 \times V = 0.03 \times 36 = 1{,}08$ cubic metres

$\therefore$ associated number of bricks in $\delta V = 450 \times 1.08 = 486$

But the cost of bricks = Rs.1100 per thousand = Rs.1.10 per brick.

$\therefore$ the approximate error in the cost = 486×1.10 = Rs.534.6

Thus the approximate error in the cost is Rs.534.60

>> If the triangle ABC is inscribed in a circle of radius r and if a, b, c respectively denotes the sides opposite to the angles A, B, C we have the sine rule (*formula*) given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

or $\quad a = 2r\sin A,\ b = 2r\sin B,\ c = 2r\sin C$

$\Rightarrow \quad \delta a = 2r\,\delta(\sin A),\quad \delta b = 2r\,\delta(\sin B),\quad \delta c = 2r\,\delta(\sin C)$

ie., $\quad \delta a = 2r\cos A\,\delta A,\quad \delta b = 2r\cos B\,\delta B,\quad \delta c = 2r\cos C\,\delta C$

or $\quad \dfrac{\delta a}{\cos A} = 2r\,\delta A,\ \dfrac{\delta b}{\cos B} = 2r\,\delta B,\ \dfrac{\delta c}{\cos C} = 2r\,\delta C$

Adding all these results we get,

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2r(\delta A + \delta B + \delta C) = 2r\,\delta(A + B + C)$$

But $\quad A + B + C = 180 = \pi$ radians $=$ constant.

$\Rightarrow \quad \delta(A + B + C) = \delta(\text{constant}) = 0$

Thus $\quad \dfrac{\delta a}{\cos A} + \dfrac{\delta b}{\cos B} + \dfrac{\delta c}{\cos C} = 0$

>> Given two sides and the included angle, to compute the thrid side we have to use the cosine formula in the standard form :

$$a^2 = b^2 + c^2 - 2bc\cos A$$

By data, $b = 60,\ c = 30,\ \hat{A} = 65^0\,24'$

But $\quad 1^\circ = 60' \Rightarrow 24' = \dfrac{24}{60} = 0.4^\circ \quad \therefore \quad \hat{A} = 65.4^\circ$

Now $\quad a^2 = 60^2 + 30^2 - 2(60)(30)\cos 65.4^\circ = 3001.389148$

$\therefore \quad a = 54.785$

Now by taking $\hat{A} = 65^\circ$ we get,

$$a^2 = 60^2 + 30^2 - 2(60)(30)\cos 65^\circ = 2978.574258$$

$\therefore$ the new $a = a_0\,(say) = 54.576$

Hence the absolute error in 'a' = $|a - a_0| = 0.209 \approx$ 0.21

[illegible]

>> Let x denote the area of the triangle ABC.

$\therefore \quad x = \dfrac{1}{2} bc \sin A$, by data.

$\Rightarrow \quad \log x = \log(1/2) + \log b + \log c + \log(\sin A)$

$\Rightarrow \quad \delta(\log x) = \delta[\log(1/2)] + \delta(\log b) + \delta(\log c) + \delta \log(\sin A)$

ie., $\quad \dfrac{\delta x}{x} = 0 + 0 + 0 + \dfrac{1}{\sin A}\cos A\,\delta A = \cot A\,\delta A$

ie., $\quad \dfrac{\delta x}{x} = \cot A\,\delta A \qquad \ldots (1)$

$\delta A = 25'$ by data. or $\delta A = \dfrac{25'}{60} = \dfrac{5^0}{12} = \dfrac{5}{12}\,\dfrac{\pi}{180}$ radians

ie., $\quad \delta A = \dfrac{\pi}{432}$ radians.

Substituting this value in (1) we get,

$$\frac{\delta x}{x} = \cot 52^\circ \cdot \frac{\pi}{432} \text{ or } \frac{\delta x}{x} \times 100 = \cot 52^\circ \cdot \frac{\pi}{432} \times 100$$

$\therefore \quad \dfrac{\delta x}{x} \times 100 = 0.5682 \approx 0.57$

Thus the percentage error in the area = 0.57

>> Area of the circle $(A) = \pi r^2$

$\Rightarrow \quad \log A = \log \pi + 2 \log r$

$\Rightarrow \quad \delta(\log A) = 0 + 2\delta(\log r)$ *ie.,* $\dfrac{\delta A}{A} = 2\dfrac{\delta r}{r}$

But $\quad \delta r = 1$ mm by data or $\delta r = 0.1$ cm and $r = 50$

$\therefore \quad \dfrac{\delta A}{A} = \dfrac{2 \times 0.1}{50} = 0.004$

or $\quad \dfrac{\delta A}{A} \times 100 = 0.004 \times 100 = 0.4$

Hence the percentage error in the area = 0.4

EXERCISES

1. Find the percentage error in calculating the area of a rectangle when an error of $+0.5\,\%$ is made while measuring the sides.
2. Find the percentage error in calculating the area of an ellipse when an error of $-3\,\%$ is made while measuring the major axis and an error of 2% is made while measuring the minor axis.
3. Find the percentage error in the volume of a right circular cone when an error of 2% is committed while measuring its height as well as the radius of its base.
4. Find the percentage error in calculating the volume and the surface area of the sphere due to an error of 0.75% in the radius of the sphere.
5. The diameter and the altitude of a right circular cylinder are measured as 4 cms and 6 cms respectively. If the possible error in each of the measurements is 0.1 cms, show that the error in the volume and the lateral surface are respectively $1.6\,\pi$ and π.
6. Find the percentage error in calculating the area of a triangle due to an error of 2% and 1% respectively in the base and the altitude.
7. In the relation $PV = RT$, find approximately the change in P correct to two decimal places as T changes from 500 to 503, V from 15.20 to 15.25 given that $P = 4000$ and R is a constant.
8. If $T = 1/2 \cdot mv^2$, find approximately the changes in T as m changes to 50.5 from 50 and V to 1495 from 1500.

9. In a triangle ABC the side BC and the angle opposite to it remain constant and the other sides and angles vary slightly, show that

$$\frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$

10. A triangle ABC has $AB = 70$ cms, $BC = 40$ cms $\hat{B} = 64^\circ\, 12'$. Find the error in the computation of the side AC by approximating $\hat{B}$ to 65°.

1. 1	2. -1	3. 6	4. 2.25, 1.5
6. 3	7. 10.85	8. 187500	10. 0.55